THE ABCD-HANKEL TRANSFORMATION IN TWO-DIMENSIONAL FREQUENCY-DOMAIN WITH POLAR COORDINATES

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Abstract
This work is devoted to a theoretical study of the ABCD-Hankel transformation. As the frequency-domain is very important in optics, we investigate a novel general Collins formula in two dimensions with polar coordinates. We also investigate the direct relationship between input and output spatial frequency spectra of a light field in this frame system in the case of a Fourier, Fresnel and fractional Fourier (FRT) transformations for two families of beams: Bessel-Gaussian and Bessel-modulated Gaussian beams.

Keywords: ABCD-Hankel; Transformation; Collins Formula; Fourier; Fresnel; Fractional Fourier.

1. Introduction
One of the most basic problems in optics is the determination of the propagation characteristics of beam waves in a paraxial optical system. In 1970, Collins [1] has found an important relationship between input and output spatial frequency spectra of a light field through an optical system characterized by A, B, C and D elements of the ray transfer matrix of the system, which is called Collins formula. We know that the ABCD law [2-5] governs the propagation and transformation of Gaussian, Hermite-Gaussian, Laguerre-Gaussian, non-Gaussian and nonspherical light beams through paraxial systems.

Earlier, Zalevsky et al [6] have provided some analytical tools on the ABCD-Bessel transformation by extending the 1-D transformation to a 2-D one, by studying some special cases. In the same year, Liu et al [7] have proposed a Collins formula in frequency-domain in two dimensions with Cartesian coordinates. This work still available for a class of beams that can be expressed in terms of Cartesian coordinates. However, in our knowledge the ABCD-Hankel transformation, which is a Bessel transformation, in two-dimensional frequency-domain with polar coordinates has not been treated before. This study is very important when the propagation of rotational symmetric beams is in need, which is the case for Bessel-Gaussian and Bessel-modulated Gaussian beams.

As a follow up of previous research, we will attempt to deduce the amplitude distribution at the output plane of a paraxial optical system, characterized by ABCD matrix, illuminated by a light field, in frequency-domain with polar coordinates. On the other hand, we know that the Hankel transformation embraces several other optical transformations, so we determine the corresponding Collins formula for several cases as Fourier, Fresnel, FRT and RSOS transformations.

2. Collins formula in a two dimensional space domain with the polar coordinates
We present in this section the basic theory of the ABCD-Hankel transformation derived from the Collins formula. For this, we consider an ABCD optical system illuminated by a light field represented by a transverse amplitude distribution $u_1(x_1, y_1)$ at the input plane and by $u_2(x_2, y_2)$ at the output plane. In this work, we consider that the input and output planes are embedded into the same optical medium. So, the determinant $BC - AD$ of the corresponding ABCD matrix should be unity. In a space domain, the generalized Huygens-Fresnel integral or the Collins formula for one transversal direction of an orthogonal system relates the output field to the input one, and the ABCD elements can be written as [1]

$$u_2(x_2, y_2) = C' \iint u_1(x_1, y_1) \exp\left(\frac{i\pi}{\lambda B} \left(\frac{AB}{2} + D(x_1^2 + y_1^2) - 2(x_1x_2 + y_1y_2)\right)\right) dx_1 dy_1, \quad (1)$$

where $k$ is the wave number and $C' = \frac{i}{\lambda B}$. With polar coordinates, one finds that the relationship between $u_1(\rho, \psi)$ and $u_2(r, \phi)$ is given by [7]
\[ u_z(r, \phi) = C' \exp \left( \frac{\pi D}{\lambda B} r^2 + n(\phi + \pi) \right) \times H_n \left[ \chi(\rho) \exp \left( \frac{i \pi A}{\lambda B} \rho^2 \right) \right] \times \exp \left( \frac{i \pi A}{\lambda B} \rho^2 \right) d\rho, \]  

(6)

is the Hankel transform of \( \chi(\rho) \exp \left( \frac{i \pi A}{\lambda B} \rho^2 \right) \).

If we apply the Parseval equality [8], with \( \nu > -1/2 \)

\[ \int_0^\infty u^*_t(v) u(v) dv = \int_0^\infty t f(t) g(t) dt, \]  

(7)

on Eq. (2), one obtains the \( C' = -\frac{i}{\lambda B} \) expression, which insures the energy conservation. We can apply Eq. (5) on several transformations embraced by the ABCD-Hankel. The definitions of the special cases of the ABCD-Hankel transformation as the Fourier, the Fresnel and the FRT transformations are listed in Table 1.

In the case of the FRT transform, which is in fact an extension of the conventional Fourier transformation to the fractional order, \( f \) is a scaling factor and the angle \( \phi \) is related to the fractional index \( p \) by \( \phi = \frac{p \pi}{2} \). Except in the case where the phase shift is \( \phi/2 \), this transform describes, in paraxial approximation of the diffraction theory, the evolution of the complex field amplitude during propagation through a quadratic refractive index medium [9]. The result corresponding to each transformation is given in Table 2.

Table 1: Definitions of the three special cases of the ABCD-Hankel transformation.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Fourier</th>
<th>Fresnel</th>
<th>FRT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A = D = 0, C = -1/f ) and ( B = f )</td>
<td>( B = z, C = 0 ) and ( A = D = 1 )</td>
<td>( B = f \sin \phi, C = -\sin \phi/\Gamma ) and ( A = D = \cos \phi )</td>
<td></td>
</tr>
</tbody>
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Table 2: The transverse amplitude distribution \( u_2(x_2, y_2) \) at the output plane for the three considered cases.

In the following, we will deduce a formula, which describes the relationship between the input and output angle spectra of a paraxial optical system. On the other hand, we will focus our attention on the above transformations in the context of two-dimensional functions, described in polar coordinates.
3. Derivation of ABCD law in the frequency space in term of polar coordinates

We know that, in this domain, the input and output angle spectra are respectively given by

\[ A_1(v_{s1}, v_{v1}) = \left[ \int u_i(x_1, y_1) \exp[-2\pi i (v_{s1} x_1 + v_{v1} y_1)] dx_1 dy_1 \right], \quad (8a) \]

and

\[ A_2(v_{s2}, v_{v2}) = \left[ \int u_2(x_2, y_2) \exp[-2\pi i (v_{s2} x_2 + v_{v2} y_2)] dx_2 dy_2 \right]. \quad (8b) \]

These angle spectra are reliable by Collins formula in frequency domain by [7]

\[ A_2(v_{s2}, v_{v2}) = i \frac{\lambda}{C} \left[ \int A_1(v_{s1}, v_{v1}) \exp\left[-i \frac{\pi \lambda}{C} \right] \right. \]

in the frequency domain for some special cases.

The second step consist of the is to evaluation of \( A_r(\rho, \varphi) \) in frequency domain in polar coordinates by the use of Eq. (9) and Eq. (10a). With the help of the following variable change

\[ v_{s2} = r_{v2} \cos \theta_{v2} \quad \text{and} \quad v_{v2} = r_{v2} \sin \theta_{v2}, \]

Eq. (9) can be written as

\[ A_2(r_{v2}, \theta_{v2}) = i \frac{\lambda}{C} \left[ \int A_1(r_{v1}, \theta_{v1}) \exp\left[-i \frac{\pi \lambda D}{C} r_{v1}^2 \right] \right] \]

where

\[ \chi(r_{v1}) = 2\pi \int_0^\infty \chi(\rho) J_n(-2\pi r_{v1}) \rho \, d\rho. \quad (13) \]

The use of Eq. (4) and Eq. (12) yields

\[ A_2(r_{v2}, \theta_{v2}) = 2\pi(-1)^n \exp\left[i\theta_{v2} - \pi \lambda z r_{v2}^2 \right] \]

in the frequency-domain for some special cases.

**Table 3:** Like Table 2, but for a rotational symmetric input beam.

| Table 4: The output angle spectra in the frequency-domain for some special cases. |
In Table 4, we summarize the different relationships of several transformations. Note that Eq. (16) is valid for any beam characterized by the radial function $\chi$ and can be used to study in the frequency-domain any transformation of a considered beam through an ABCD axisymmetric optical system.

4. Applications

In this section, we will discuss the propagation in the frequency-domain of Bessel-Gaussian and QBG light beams and provide new formulas of some special cases as Fourier, Fresnel and FRT transformations. So, Eq. (16) can be written as

$$A_2(r_{v_2}, \theta_{v_2}) = 2\pi \frac{\lambda}{\gamma} \exp\left(-i \frac{\lambda A^2 v_2^2}{r_{v_2}}\right) e^{i \theta_{v_2}} \times \int_0^\infty \left(\chi'(r_\nu) e^{-\gamma^2 r_\nu^2} \right)^2 J_0(\beta r_\nu) r_\nu^2 d r_\nu$$

where

$$\beta = \frac{2\pi \lambda}{\gamma} r_{v_2}, \hspace{1cm} (19a)$$

and

$$\gamma^2 = \frac{i \pi \lambda D}{\gamma} \hspace{1cm} (19b)$$

Eq. (18) is available for $C \neq 0$. If $C = 0$ and by using the following simple input-output relationship given in Ref. [7]

$$A_2(r_{v_2}, \theta_{v_2}) = \frac{1}{D} \exp\left(-\frac{i n B}{D} \left[r_{v_2} - \frac{r_{v_1}}{D} \right] \right) A_1^0 \left[r_{v_1} - \frac{r_{v_1}}{D} \right], \hspace{1cm} (20a)$$

one obtains with the same procedure as above

$$A_2(r_{v_2}, \theta_{v_2}) = 2\pi (-1)^n \exp\left(i n \theta_{v_2} - \pi \lambda z r_{v_2}^2 \right) \times \int_0^\infty \left(\chi'(r_\nu) J_n(2\pi r_\nu^2 \rho) \right) d \rho. \hspace{1cm} (20b)$$

In the following, we will apply Eq. (18) in the case of the two families of beams: Bessel-Gauss and QBG.

4.1 Bessel-Gauss beams

This family of beams can be expressed in terms of the Bessel function $J_n$ of the first kind and nth order [10-12]. The function $\chi$, for these beams, is defined by

$$\chi(\rho) = J_n(\alpha \rho) \exp\left(-u^2 \rho^2 \right), \hspace{1cm} (21a)$$

where

$$u^2 = \frac{1}{\omega_0^2}, \hspace{1cm} (21b)$$

$$\alpha = k \sin \theta, \hspace{1cm} (21c)$$

and

$$\omega_0^2 = 2\gamma R \gamma. \hspace{1cm} (21d)$$

In these equations, $k = 2\pi/\lambda$ is the wave number of the field and the parameters $\theta, z$, and $\omega_0$ are respectively the cone angle of the ideal non-apodized Bessel field (in the paraxial approximation), the Rayleigh range and the spot size of the fundamental Gaussian mode. If $\omega_0 \to \infty$ one obtains a pure Bessel field and if $\alpha \to 0$ an ordinary Gaussian beam is established.

With the help of Eq. 6.633.2 of Ref. [10], one obtains

$$\chi'(r_\nu) = (-i)^n \frac{\pi}{u^2} \exp\left(-\frac{\alpha^2 + \xi^2}{4u^2} \right) J_n\left(\frac{\alpha \xi}{u^2} \right). \hspace{1cm} (22)$$

where $I_n$ is the modified Bessel function of order $n$ given by $I_n(x) = (-i)^n J_n(ix)$ and $\xi = 2\pi r_\nu$.

So, Eq. (18) becomes

$$A_2^{BG}(r_{v_2}, \theta_{v_2}) = A_{BG}(\theta_{v_2}) \exp\left(-R_{BG} r_{v_2}^2 \right) \times J_n(h_{BG} r_{v_2}), \hspace{1cm} (23)$$

where

$$A_{BG}(\theta_{v_2}) = \frac{i \pi^2 \omega_0^2}{\Gamma C} \left(\frac{\pi \lambda}{\Gamma C} + i A \right), \hspace{1cm} (24a)$$

and

$$R_{BG} = \frac{\pi \lambda}{\Gamma C}, \hspace{1cm} (24b)$$

$$h_{BG} = \frac{\pi^2 \omega_0^2 \lambda}{\Gamma C}, \hspace{1cm} (24c)$$

with

$$\Gamma = \pi^2 \omega_0^2 + \frac{i \pi \lambda D}{\gamma}. \hspace{1cm} (24d)$$

To get Eq. (23), we have used Eq. 6.633.4 of Ref. [10]. For $C=0$, by the use of Eq. (20b) one obtains

$$A_2^{BG}(r_{v_2}, \theta_{v_2}) = (-1)^n \pi \omega_0^2 \exp\left(\text{in} \theta_{v_2} - \frac{\alpha^2 \omega_0^2}{2} \right) \times \exp\left[\left(\pi^2 \omega_0^2 + i \pi \lambda z \right) \right] J_n\left(\pi \alpha \omega_0^2 r_{v_2} \right). \hspace{1cm} (24e)$$

We give in Table 5, the expressions of $A_2^{BG}(r_{v_2}, \theta_{v_2})$ in the three cases: Fourier, Fresnel and RSOS transformations.
and \( \chi' \) is obtained, by the use of Eq. 6.651.6 of Ref. [10], as

\[
\chi'(r_\nu) = (-i)^n \frac{\pi}{\sqrt{\alpha^2 + u^2}} \exp \left( -\frac{\pi^2 u^2}{\alpha^2 + u^4 r_\nu^2} \right) \times J_{\frac{n}{2}} \left( \frac{\pi^2 \alpha}{\alpha^2 + u^4 r_\nu^2} r_\nu^2 \right).
\]

Eq. (18) becomes in this case

\[
A_{QBG}^{BG} (r_\nu, \theta_\nu) = A_{QBG} (\theta_\nu) \exp \left( -\frac{\pi^2 \alpha}{\alpha^2 + u^4 r_\nu^2} r_\nu^2 \right),
\]

where

\[
A_{QBG} (\theta_\nu) = \frac{4i\pi^2 \alpha \omega_0^2}{C \sqrt{\sin^2 \phi + \tau^2}} \left( \frac{1 + \alpha^2 \omega_0^4}{1 + \alpha^2 \omega_0^4} \right),
\]

and

\[
h_{QBG} = \frac{\tau \epsilon^2}{\eta^2 + \tau^2},
\]

with

\[
\epsilon = 2\pi \frac{\lambda}{C}, \quad \tau = 4\alpha \omega_0^2 s,
\]

\[
\eta = 4 \left( s + \frac{3D}{C} \right),
\]

where

\[
s = \frac{\pi^2 \omega_0^2}{1 + \alpha^2 \omega_0^2}.
\]

The expressions of the important parameters \( \epsilon \) and \( \eta \) which are needed to evaluate \( A_{QBG}^{BG} (r_\nu, \theta_\nu) \) for Fourier, FRT and RSOS transformations are:

(\( \epsilon, \eta \)) = \left( -2\pi \lambda f, 4s, 4(s - i \frac{\pi \lambda f}{\sin \phi} \frac{\phi}{1 + \frac{\pi}{\lambda f} \tan \phi} \right),
\]

and \( \left( -2\pi \lambda f, 4(s - i \frac{\pi}{\lambda f} \tan \phi) \right) \), respectively. Eq. (31) is independent of transformation parameters. Eq. (28) is available for \( C \neq 0 \). In the case of \( C = 0 \), Eq. (24e) can be written as

\[
A_{QBG}^{BG} (r_\nu, \theta_\nu) = \pi(-1)^n \exp \left( i \theta_\nu^2 - \pi \lambda \alpha r_\nu^2 \right),
\]

where

\[
A_{QBG} = \omega_0 \sqrt{s}/\pi.
\]
As for the Bessel-Gaussian beams, we deduce that the output fields in the frequency-domain of the Bessel-modulated Gaussian beams have the same structure as the input field. So, this family of beams constitutes also a class of fields, in the frequency-domain, whose form is invariant after propagation in a paraxial optical system characterized by an A, B, C and D elements.

5. Conclusion

In summary, we have studied the ABCD-Hankel transformation in two-dimensional frequency-domain with polar coordinates. Our formulas show that the angle spectrum propagation obeys to a novel Collins formula expressed in the frequency-domain, which is similar to that of the complex amplitude propagation. We can deduce that the output fields in the frequency-domain have the same structure as the input Bessel-Gaussian and Bessel-modulated Gaussian fields. On the other hand, we have shown that the studied families of beams constitute a class of fields whose form is invariant after propagation in a paraxial optical system characterized by an A, B, C and D elements. Starting from the novel Collins formula, some transformations are analyzed: free space propagation, Fourier, Fresnel and FRT transformations in cases: Bessel-Gaussian and QBG.

References