PROPAGATION OF MODIFIED BESSEL-GAUSSIAN BEAMS THROUGH AN ANNULAR APERTURED PARAXIAL ABCD OPTICAL SYSTEM

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Abstract
We investigate the propagation characteristics of modified Bessel-Gaussian (MBG) beams travelling through a paraxial ABCD optical system with an annular aperture. A novel analytical expression of MBG beams intensity distribution is examined by the use of the generalized Collins diffraction integral and the expansion of the hard aperture function into a finite sum of complex Gaussian functions. From our analytical expression, the free space, circular aperture and circular black screen are also treated as particular cases. The previous results concerning the propagation of Bessel-Gaussian, Bessel and Gaussian beams through the considered optical system are found as special cases. Some numerical examples are given to simulate the propagation of MBG beams.

Keywords: Modified Bessel-Gaussian beams; Collins diffraction integral; Annular aperture; Circular aperture; Circular black screen; Paraxial ABCD optical system; Hard aperture function.

1. Introduction
Since the early years of discovering laser beams, their propagation, which is an important and useful property, has been subject of continuing research interest. In this work, we will investigate the propagation characteristics of MBG beams travelling through a paraxial ABCD optical system with an annular aperture. The considered field is written down in closed form for Gaussian fields modulated by a summation of modified Bessel functions. Several papers have been devoted to the study of these beams in various optical systems [1-6] and in turbulence [7].

Up to now, the propagation of a MBG beam through a paraxial ABCD optical system with an annular aperture, which represents the most general case, to our knowledge, has not been examined elsewhere. We have elaborated in this work a solution of this problem.

However, when investigating the radiation propagation through an ABCD optical system, one usually uses the so-called Collins diffraction integral formula [8]. The later, known as the generalized Huygens-Fresnel diffraction integral, which gives the relationship between the input and output complex amplitude distributions of the electromagnetic fields, is investigated to optional resonators, optical filtering, holography, optical-beam waveguides and other physical devices. So, by means of the Collins formula, and using the similar way as for Laguerre-Gaussian beams [9-10] and Bessel-modulated Gaussian beams [11], we will formulate an approximate analytical expression for the receiver plane intensity of the MBG beam travelling across the annular aperture paraxial ABCD optical system. From this investigation, we will derive the cases of unapertured, apertured and black screen as particular cases of the more general case of annular aperture. Also, from our generalized analytical expressions, the previous results [11] about the propagation of Bessel-Gaussian beams through a paraxial ABCD optical system with an annular aperture and all their special cases will be deduced.

This paper is outlined as follows. In Section 2, we derive the approximate closed-form expression of the MBG beams propagating through a paraxial ABCD optical system with an annular aperture by expanding the circ function into a finite sum of complex Gaussian functions. In Section 3, we calculate analytically the closed-form expressions for some particular cases. To illustrate the propagation properties, some numerical simulations are given in Section 4. And a conclusion is given in the last section.

2. Transformation of MBG beams by an annular apertured paraxial ABCD optical system
Our analysis starts by defining the field of MBG beam at the input plane, \( E(\rho_0, \phi_0, z = 0) \), of a first-order optical ABCD system (A, B, C and D
are the coefficients of the transfer matrix characterizing the optical system) as

$$E(r_0, \varphi_0, z = 0) = \exp(-k\alpha r_0^2) \sum_n A_n \exp(-i\alpha \varphi_0) I_n(a_y r_0)$$  \(1\)

\((r_0, \varphi_0, z = 0)\) represents the cylindrical coordinates at \(z=0\). The term \(\exp(-k\alpha r_0^2)\) constitutes the Gaussian part and \(\alpha = \frac{1}{k\alpha_x^2} + \frac{i}{(2F_x)}\), where \(\alpha_x\) and \(F_x\) refer, respectively, to radial Gaussian source and focusing parameter. \(k = \frac{2\pi}{\lambda}\) is the wave number with \(\lambda\) being the wavelength. The Bessel part is governed by a summation over modified Bessel functions of order \(n\). \(A_n\) and \(a_y\) are, respectively, the amplitude coefficient and the width parameter.

In Fig. 1, we illustrate in the source plane, a collection of the incident multiple MBG beams intensity at different orders and with various amplitude coefficients. These figures show some illustrations, which are arranged to be in cartesian coordinates in mm, containing more than one beam in the summation. The vertical axis corresponds to normalized intensity of the considered beams.

Let us now consider the Collins formula for investigating the output field distribution,

$$E(r, \varphi, z) = \frac{ik}{2\pi B} \int_0^\infty \int_0^{2\pi} E(r_0, \varphi_0, z = 0) \exp \left\{ -\frac{ik}{2B} \left[ dr_0^2 - 2rr_0 \cos(\varphi - \varphi_0) + Dr^2 \right] \right\} r_0 dr_0 d\varphi_0,$$  \(2\)

\((r, \varphi, z)\) denote the cylindrical coordinates in output plane at distance \(z\) from source plane, \(a\) and \(b\) are the outer and inner radius of the annular aperture, respectively. For the sake of simplicity, in this last equation, an unimportant phase factor is omitted. In Fig. 2, we illustrate the propagation of the considered beam through an annular apertured paraxial ABCD optical system.

To transform the finite integral of Eq. (2) into an infinite one, we will use the following technique of hard aperture functions

$$E(r, \varphi, z)$$, through the annular apertured paraxial ABCD optical system, which is given by

$$A_{pa}(r) = \begin{cases} 1 & \text{for } |r| \leq a, \\ 0 & \text{for } |r| > a \end{cases},$$  \(3a\)

and

$$A_{pb}(r) = \begin{cases} 1 & \text{for } |r| \leq b, \\ 0 & \text{for } |r| > b \end{cases}.$$  \(3b\)

It has been shown that the hard aperture functions can be expanded into finite sums of complex Gaussian functions as [12]

$$A_{pa}(r_0) = \sum_{k=1}^{M} A_k \exp \left( -\frac{B_k r_0^2}{2} \right),$$  \(4a\)
and

\[ A_{pb}(r_0) = \sum_{g=1}^{M} A_g \exp\left( -\frac{B_g}{b^2} r_0^2 \right), \]

(4b)

where \( A_b, B_b, A_g, \) and \( B_g \) are, respectively, the expansion and Gaussian coefficients. In the following, we will take \( M=10 \) [12].

**Figure 2:** MBG beam propagating through an annular apertured paraxial ABCD optical system.

By introducing Eq. (1) into Eq. (2) and by recalling the integral formulae [13-14]

\[ \int_0^{2\pi} e^{-i\alpha x} \frac{e^{-i\beta}}{\sqrt{2\pi}} \cos(\gamma x) \, dx = \frac{1}{2\alpha} \exp\left( -\frac{\beta^2 - \gamma^2}{4\alpha^2} \right) \frac{\beta}{2\alpha}, \]

(6)

and

\[ \int_0^{2\pi} e^{-i\alpha x} I_n(\beta x) J_n(\gamma x) \, dx = \frac{\beta}{2\alpha} \frac{J_n(\beta x)}{I_n(\beta x)}, \]

(7)

with \([\text{Re } \alpha > 0, \text{ Re } \beta > 0]\),

the output field distribution on the receiver plane, situated at one axial distance \( z \) away from source plane, becomes, after tedious integral calculations,

\[ E(r, \varphi, z) = \frac{ik}{B} \exp\left( -\frac{ikD}{2B} r^2 \right) \sum_{n} A_n \exp\left( \frac{in(\pi - \varphi)}{2} \right) \sum_{a=1}^{M} A_a \sum_{b=1}^{N} \frac{\alpha + \beta}{4\gamma_{a,b}} J_n \left( \frac{\alpha + \beta}{2\gamma_{a,b}} \right), \]

(7)

with

\[ \gamma_{a,b} = \frac{\alpha}{\alpha} + kx + \frac{ka}{2B}, \]

(8a)

\[ \gamma_{b,g} = \frac{\beta}{\beta} + kx + \frac{kb}{2B}, \]

(8b)

and

\[ \beta = \frac{k r}{B}. \]

(8c)

Eq. (7) represents the main result of this study and permits an arbitrary MBG beam to be propagated in closed-form through a real or complex ABCD optical system with an annular aperture.

To examine the validity of this result, we will prove that from Eq. (7), one obtains the main result of Ref. [11] concerning the propagation of Bessel-Gaussian beam through the considered optical system. So, by taking account of the following changing constants

\[ a_B = iaB, \]

(9a)

\[ N = n = m, \]

(9b)

and

\[ A_n(\gamma) = C_0, \]

(9c)

Eq. (7) becomes

\[ E(r, \varphi, z) = \frac{ik}{B} C_0 \exp\left( -i\varphi \right) \exp\left( -\frac{ikD}{2B} r^2 \right) \times \sum_{a=1}^{M} A_a \exp\left( -\frac{\alpha^2 + \beta^2}{4\gamma_{a,B}} \right) J_n \left( \frac{ia\beta}{2\gamma_{a,B}} \right) \]

\[ \times \sum_{b=1}^{N} \frac{A_b}{2\gamma_{B,b}} J_n \left( \frac{ia\beta}{2\gamma_{B,b}} \right) \]

\[ -\sum_{g=1}^{M} A_g \exp\left( -\frac{\alpha^2 + \beta^2}{4\gamma_{B,g}} \right) J_n \left( \frac{ia\beta}{2\gamma_{B,g}} \right). \]

(10)

This equation is the result of the study of Mei et al. [11]. When \( \omega_0 \to \infty \), it can be reduced to the propagation equation through an annular apertured paraxial optical ABCD system of a pure Bessel beam and of a Gaussian one when \( \alpha = 0 \) and \( m = 0 \). And we will examine some special cases of the annular apertured paraxial ABCD optical system in the next section.

3. Special cases

In the above section, we have demonstrated that the MBG beams can be propagated through an ABCD optical system in a closed-form. To examine the propagation details of this family beams, we study the deduction from Eq. (7) of the corresponding closed-form for the unaperture,
circular aperture and circular black screen as particular cases of the annular aperture.

3.1. Unaperture case

For this case, we have $b \to 0$ and $a \to \infty$. So, Eqs. (8) reduce to

$$\gamma_{a,b} = k a + \frac{ik b}{2 B} = \gamma, \quad (11a)$$

and

$$\gamma_b \to \infty. \quad (11b)$$

After insertion of these parameters, Eq. (7) becomes

$$E(r,\varphi,z) = \frac{ik B}{C_0} \exp\left(-\frac{ik D r^2}{2B}\right) \sum_{m=1}^{M} A_h \exp\left[-\frac{\alpha^2 + \beta^2}{4 \gamma}\right] J_m\left(\frac{i a \beta r}{2 \gamma}\right)$$

This equation denotes the output field distribution of the MBG beams through an unapertured paraxial optical ABCD system.

Inserting Eqs. (9) into Eq. (12), one obtains

$$E(r,\varphi,z) = \frac{ik B}{C_0} \exp(-imp) \exp\left(-\frac{ik D r^2}{2B}\right) \sum_{m=1}^{M} A_h \exp\left[-\frac{\alpha^2 + \beta^2}{4 \gamma}\right] J_m\left(\frac{i a \beta r}{2 \gamma}\right)$$

The above equation is the same as Eq. (13) in Ref. [11], which describes the output field distribution of a Bessel-Gaussian beam passing through a paraxial ABCD optical system without aperture.

3.2. Circular aperture case

This case corresponds to $b \to 0$ and $a \neq 0$, and the output field distribution of a MBG beam travelling across a paraxial optical ABCD system with a circular aperture of radius $a$ is given by

$$E(r,\varphi,z) = \frac{ik B}{C_0} \exp\left(-\frac{ik D r^2}{2B}\right) \sum_{m=1}^{M} A_h \exp\left[-\frac{\alpha^2 + \beta^2}{4 \gamma_{a,h}}\right] J_m\left(\frac{i a \beta r}{2 \gamma_{a,h}}\right)$$

If we take into account Eqs. (9), this latter equation becomes

$$E(r,\varphi,z) = \frac{ik B}{C_0} \exp(-imp) \exp\left(-\frac{ik D r^2}{2B}\right) \sum_{m=1}^{M} A_h \exp\left[-\frac{\alpha^2 + \beta^2}{4 \gamma_{a,h}}\right] J_m\left(\frac{i a \beta r}{2 \gamma_{a,h}}\right)$$

Eq. (15) is the same as Eq. (15) of Ref. [11] which corresponds to the propagation of Bessel-Gaussian beam through a paraxial ABCD optical system with a circular aperture.

3.3. Circular black screen

When $a \to \infty$ and $b \neq 0$, by using Eq. (7) the output field distribution on the receiver plane passing the considered optical system with a circular black screen is given by

$$E(r,\varphi,z) = \frac{ik B}{C_0} \exp\left(-\frac{ik D r^2}{2B}\right) \sum_{m=1}^{M} A_h \exp\left[-\frac{\alpha^2 + \beta^2}{4 \gamma}\right] J_m\left(\frac{i a \beta r}{2 \gamma}\right)$$

$$\times \left\{ \frac{A_h}{2 \gamma} \exp\left[\frac{a^2 - \beta^2}{4 \gamma}\right] J_{m\alpha}\left(\frac{a \beta r}{2 \gamma}\right) \right\}$$

Replacing Eqs. (9) in the above equation, one obtains Eq. (16) of Ref. [11] corresponding to the output field distribution of a Bessel-Gaussian beam passing through the considered system with a circular black screen. So, in this case, the output field is given by

$$E(r,\varphi,z) = \frac{ik B}{C_0} \exp(-imp) \exp\left(-\frac{ik D r^2}{2B}\right) \sum_{m=1}^{M} A_h \exp\left[-\frac{\alpha^2 + \beta^2}{4 \gamma_{b,g}}\right] J_m\left(\frac{i a \beta r}{2 \gamma_{b,g}}\right)$$

4. Numerical simulations and discussions

The aim of this section is to analyse the propagation of MBG beams through an apertured annular paraxial optical ABCD system and also through a circular aperture in free space, using respectively the above analytical expressions (7) and (14). In the simulations, we choose $M=10$ and the coefficients $A_a$, $B_a$, $A_g$, and $B_g$ of a hard aperture as mentioned in Ref. [12]. Fixed Gaussian size, collimated beams and a single wavelength are employed in simulations. This means that $\alpha_g = 0.5 mm$, $F_s \to \infty$ and $\lambda = 0.6328 \mu m$. The width parameter is fixed in $a_B = 24810 m^{-1}$ and the transfer matrix elements are $A=1$, $B=2$, $C=0$ and $D=1$.

As a first example, by using Eq. (7), Fig. 3 gives the intensity of MBG travelling an annular aperture in free space for various Fresnel numbers $F_w = 10$, 1 and 0.1 (where $F_w = 1/(2 \pi a z)$). The outer and inner radius chosen are, respectively, $a = 0.2 mm$ and $b = 0.05 mm$. Figs. 3-a present the receiver intensity of the cited beams for two single orders. We remark that the lobes have vanished bit by bit with the decrease of $F_w$ (i.e. with the increase of $z$). The receiver intensity of the multiple even and odd orders MBG beams are performed, respectively, in Figs. 3-b and 3-c. Finally, in Fig. 4, as a particular case of apertured annular, we present the radial intensity of MBG beams passing by a circular aperture (with $b \to 0$...
and \( a \neq 0 \) for different values of \( F_w \) (the same values chosen above).

From these figures, one can see that for the same order, the beam profile at the plane \( F_w = 1 \) is almost the same as that at the plane \( F_w = 0.1 \). This profile changes considerably at the plane \( F_w = 10 \) and the profile is accompanied by the apparition of other lobes. In all graphs, for the same order and same paraxial optical system (annular or circular aperture), we can see that the receiver intensity conserves his profile with an amplification in space following \( r \) by respecting an inverse of ratio of \( F_w \).

**Figure 3:** Receiver MBG beams intensity distributions travelling across an annular apertured free space for various amplitude coefficients \( A_n \), and different Fresnel numbers \( F_w (=10, 1 \text{ and } 0.1) \) with inner radius \( b = 0.05 \text{ mm} \) and outer radius \( a = 0.2 \text{ mm} \) at (a) a single order, and (b) a multiple odd orders.
Figure 4: Receiver MBG beams intensity distributions travelling across a circular apertured free space for various Fresnel numbers $F_w (=10, 1$ and $0.1)$, and with radius of circular aperture $b=0.05mm$ for various amplitude coefficients $A_n$ at (a) a single order, (b) a multiple odd orders, and (c) a multiple even orders.
5. Conclusion

In sum, starting from Collins diffraction formula, which gives the relationship between the input and output complexes amplitudes distributions of the Lasers fields, and by using the expansion of the hard aperture function into a finite sum of complex Gaussian functions, we have derived the receiver field distribution of propagation of MBG beams through an annular apertured paraxial ABCD optical system. Our finding could be reduced to the cases of unaperture or circular aperture, or circular black screen. Some numerical simulations have been performed to illustrate the propagation of this beams family. Finally, the propagation of a Gaussian, Bessel and Bessel-Gaussian beams through an annular apertured paraxial ABCD optical system could be regarded as special cases of its MBG beams.

References