PARAXIAL PROPAGATION OF VECTOR MATHIEU-GAUSS BEAMS

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Abstract
Based on the expansion of the scalar Mathieu-Gauss beam in terms of Bessel-Gauss beams we give a general expression of vector Mathieu-Gauss (vMG) beams in cylindrical coordinates. Furthermore, the analytical expression of transverse vector Mathieu-Gauss beams through any axisymmetric optical ABCD system is derived by using the Collins diffraction formula. Particularly, the propagation of the vector Mathieu-Gauss beam in free space and through a simple lens system are illustrated numerically and discussed.

Keywords: Vector Mathieu-Gauss beams; Bessel-Gauss beams, Mathieu beams; Collins formula.

1. Introduction
The major studies on the laser beams consider the scalar wave equation and the paraxial approximation to describe the character of the fields radiation associated with these beams. This treatment is adequate for applications that do not involve the polarization vector. For applications as microscopy, lithography, material processing, electron acceleration, optical trapping among others [1-4], the knowledge of beam polarization states is of major concern. For that reason, some pioneering works have been investigated for finding vector solution of Maxwell’s wave equations. Starting from these equations, a power expansion procedure of the electric field vector was developed by Lax et al. in 1976 [5]. Lax perturbative series expansion show that in zeroth-order the vector field is purely transverse, but in the next order correction a small longitudinal component of vector field appears. In the same way, Davis [6] has reproduced the results of Lax et al. [5] for the zeroth and first order fields of a Gaussian beam by adopting the perturbative expansion series of the magnetic vector potential. In practical domain, Erdogan et al [7] have realized, since 1994, a first near infrared, circularly symmetric, azimuthally polarized laser beam. Later, based on this fact, Hall et al. [8-10] developed, in many original papers, an approach based on the scalar wave equation to obtain theoretically vector Bessel-Gauss beam and particularly the azimuthally polarized Bessel-Gauss beam. The free space propagation and the focusing properties of this last beam have been treated in detail [10]. Recently, Brandres and Gutiérrez-Vega [11] have used the Lax expansion and the vector solution of the three dimensional Helmholtz equation to demonstrate that vector Helmholtz-Gauss (vHgZ) beams and vector Laplace-Gauss beams constitute the general families of localized beam solutions of the Maxwell equation in the paraxial approximation. The TE and TM Gaussian vector beams, nondiffracting vector Bessel [12] beam and polarized Bessel-Gauss beams are regarded as a particular cases of vHzG beam.

This paper is aimed at studying the transformation of vector Mathieu-Gauss (MG) beams through an ABCD axisymmetric optical system. The analytical expression of these beams in terms of Bessel-Gauss beams [13-14] is given in Section 2. Their paraxial propagation equations through unapertured ABCD optical system and some numerical calculations and concluding remarks related to free propagation and focalization by a simple lens system are given in Section 3. A summary is presented in the conclusion.

2. Vector Mathieu-Gauss beams in cylindrical coordinates system
A description of vector beam propagation begins with the vector wave equation for the electric field:

\[ \Delta \mathbf{E} + k^2 \mathbf{E} = 0 \]  

where an \( \exp(iwt) \) was assumed and \( k = \omega/c \). In the paraxial limit and by using the Lax’s expansion and the scalar solution of two-dimensional Helmholtz equation (see Ref. [11]), the general expression in cylindrical coordinates of vector MG components corresponding to the first and the second families are, respectively
\[
\begin{align*}
E_{p,m}^{(1)}(\rho, \varphi, z = 0) &= \frac{\partial W_{m}^{z,\alpha}(\rho, \varphi, z = 0)}{\partial \rho} \exp \left( \frac{-\rho^2}{w_0^2} \right), \quad (2.a) \\
E_{p,m}^{(1)}(\rho, \varphi, z = 0) &= \frac{1}{\rho} \frac{\partial W_{m}^{z,\alpha}(\rho, \varphi, z = 0)}{\partial \varphi} \exp \left( -\frac{-\rho^2}{w_0^2} \right) \\
E_{r,m}^{(1)}(\rho, \varphi, z = 0) &= -\frac{2}{kw_0} \frac{\partial W_{m}^{z,\alpha}(\rho, \varphi, z = 0)}{\partial \rho} + \frac{k w_0}{k} \frac{\partial W_{m}^{z,\alpha}(\rho, \varphi, z = 0)}{\partial \rho} \exp \left( \frac{-\rho^2}{w_0^2} \right)
\end{align*}
\]

where \( w_0 \) is the Gaussian waist at the plane \( z = 0 \), \( W_{m}^{z}(\rho, \varphi, z) \) are the even (the superscripts \( e \)) and odd (the superscripts \( o \)) Mathieu modes. The integer index is \( m \geq 0 \) or \( m \geq 1 \) for even and odd modes, respectively. And \( (\rho, \varphi, z) \) denote the cylindrical coordinate. As it is known, the relationship between of Mathieu solutions and Bessel solutions of the scalar two dimensional Helmholtz equation [14] is given by

\[
\begin{align*}
W_{m}^{z,\alpha}(\rho, \varphi, z) &= \sum_{n=0}^{\infty} A_n^{(m)}(q) \cos(n \varphi) J_n(k \rho) \\
W_{m}^{z,\alpha}(\rho, \varphi, z) &= \sum_{n=0}^{\infty} B_n^{(m)}(q) \sin(n \varphi) J_n(k \rho)
\end{align*}
\]

where \( J \) is Bessel function of the first kind and \( k \) is the wave vector transverse component. \( A_n^{(m)} \), \( B_n^{(m)} \) are expansion coefficient of Mathieu functions and \( q \) is the elliptic parameter given by \( q = k^2 k_i^2 / 4 \), where \( h \) is the interfocal separation [15].

On the other hand, a general vector beam solution of Maxwell equation in paraxial regime is obtained as the superposition of the solutions of Eqs. (2) [11]. So,

\[
\vec{E} = \alpha \vec{E}^{(1)} + \beta \vec{E}^{(2)},
\]

where \( \alpha \) and \( \beta \) are arbitrary constants. The field of Eq. (4) satisfies the transverse wave equation for all \( \alpha \) and \( \beta \) and the propagation properties of vMG are governed by both solutions \( \vec{E}^{(1)} \) and \( \vec{E}^{(2)} \). Thus, after some calculations the radial and azimuthal components of the vMG beams read

\[
\begin{align*}
E_{p,m}^{(1)}(\rho, \varphi, z = 0) &= U_{p,m}^{(1)}(\rho, z = 0) \times \exp \left( \frac{-\rho^2}{w_0^2} \right) \quad (5.a) \\
E_{p,m}^{(1)}(\rho, \varphi, z = 0) &= U_{p,m}^{(1)}(\rho, z = 0) \times \exp \left( -\frac{-\rho^2}{w_0^2} \right) \quad (5.b)
\end{align*}
\]

with

\[
\begin{align*}
U_{p,m}^{(1)}(\rho, z = 0) &= E_0 \sum_{n=0}^{\infty} A_n^{(m)}(q) J_n(k \rho) T_n(k \rho) \times \phi_{p,m}(\varphi) \\
U_{p,m}^{(1)}(\rho, z = 0) &= E_0 \sum_{n=0}^{\infty} B_n^{(m)}(q) J_n(k \rho) T_n(k \rho) \times \phi_{p,m}(\varphi)
\end{align*}
\]

where \( \phi_{\rho}(\varphi) = \cos(n \varphi) \) \( (5.d) \)

\( \phi_{\varphi}(\varphi) = -\sin(n \varphi) \) \( (5.e) \)

The components of the second solutions are related to the first components by

\[
\begin{align*}
E_{p,m}^{(2)}(\rho, \varphi, z = 0) &= E_{p,m}^{(1)}(\rho, \varphi, z = 0) \\
E_{p,m}^{(2)}(\rho, \varphi, z = 0) &= -E_{p,m}^{(1)}(\rho, \varphi, z = 0)
\end{align*}
\]

In Eq. (5.c), radial components use the upper sign (\( + \)) of the (\( \mp \)) sign and the azimuthal components use the lower one (\( + \)). By taking \( a_m = -b_m \) in Eqs. (12) and (13) of Ref. [9], one can easily deduce from Eqs. (5) that vMG beam components are expressed as a sum of vector Bessel-Gauss beams components. This means that we retrieve the similar result yet found for the scalar case i.e, MG beam can be expressed as sum of Bessel-Gauss beam [13]. Likewise, one can obtain the field expression for the odd families of vMG beams by replacing in Eqs. (5.d) and (5.e) the functions \( \cos(n \varphi) \) and \( -\sin(n \varphi) \) by \( \sin(n \varphi) \) and \( \cos(n \varphi) \), respectively.

3. Transformation of vMG beams components through an axisymmetric ABCD optical system

In this last paragraph, only transverse components are considered because in zeroth-order the field is purely transverse and depend only on the transverse coordinates \( \rho \) and \( \varphi \). In order to propagate these components through an axisymmetric paraxial \( \text{ABCD} \) optical system, we use the well known Collins integral Formula [16]. The obtained field distribution at any plane located at \( z \) reads (for details see Refs. [10-11, 13])
\[ E_{[\rho,\varphi]}^{(1)}(\rho,\varphi,z=0) = \exp \left( -\frac{k_z^2 B}{2k} \right) \times U_{[\rho,\varphi]}^{(m)} \left( \frac{\rho}{\mu} \right) \times GB(\rho), \quad (6.a) \]

with \( \mu = A + iB / z_R, \)

and \( GB(\rho) \) is the fundamental Gaussian beam

\[ GB(\rho) = \exp \left( i k z \right) \exp \left( -\frac{\rho^2}{2 \rho_0^2} \right), \quad (6.c) \]

where \( z_R = k \rho_0^2 / 2 \) being the Rayleigh range of Gaussian beam and \( A, B, C \) and \( D \) are the matrix transfer coefficients.

In the following, we will use Eq. (6.a) to analyze the propagation of MG beam transverse components in some important cases. As a first example, we consider the propagation of transverse vector field components through unapertured free space. In this case the matrix transfer coefficients are: \( A = 1, B = z, C = 0 \) and \( D = 1 \). In Fig. (1), we present the evolution of the radial components of zeroth-order MG beam associated with the first solution of vector beams.

The calculation parameters are: \( q=20 \), the Gaussian waist \( w_0 = 50mm \), \( k_z = 2 \times 10^4 \text{ m}^{-1} \) and \( \lambda = 632.8mm \). As it is known, the analysis of propagation of any nondiffracting beams modulated by a Gaussian envelope, allow us to introduce the following parameters: \( \gamma = \frac{1}{2} k_z \rho_0 \) and \( z_{\text{max}} = z_R / \gamma \). The first parameter plays an important role in determining the behavior of components vMG beam upon propagation [10] and the second one give the maximal distance propagation. For the given simulation values the calculation of these parameters gives: \( \gamma = 50 \) and \( z_{\text{max}} = 248m \). Thus, one can conclude that the nondiffracting beams will impose the propagation characteristics within the range \( z \leq z_{\text{max}} \) and the Gaussian envelope will survive up to this distance [11, 13]. It is to be noted, that for the numerical calculation, we have interested in radial intensity because the azimuthal one is negligible for the first solution (Eq. (2.a)) and the inverse for second.
solution (Eq. (2.b)). On the other hand, the given values, satisfy the conditions $k_0 \gg k_w$ and $\gamma_1 \gg \gamma_2$, and then the longitudinal components are also negligible.

In the second example, we consider the propagation of the vector beam through a nonaperturing thin lens. In this case the matrix transfer coefficients are: $A=1-z/f$, $B=z$, $C=-1/f$ and $D=1$, where $f$ is the focal length of the thin lens. In Fig. (2), we present the evolution of radial intensity versus $z$ at the point $(\rho = 0.825 \text{mm}, \varphi = 0)$ of maximum transverse intensity. For the given values of calculations, $\gamma_1 \gg \gamma_2$ in both cases of Figs.(2.a) and (2.b), so the focalization properties are those of the nondiffracting beams. This fact is clearly shown in Fig. (2.b), where we have presence of two peaks of maximum intensity after and before the geometrical focus.

![Figure 2: Focalization of zeroth-order even vMG by a thin lens of focal length $f = 0.5 \text{m}$, for $q = 20$. Variation of intensity versus $z$ for $\rho = 0.825 \text{mm}$ and $\varphi = 0$ for $k_0 w_0 = 100$ and (a) $w_0 = 2 \text{mm}$ and (b) $w_0 = 10 \text{mm}$.](image)

4. Conclusion

We have studied the propagation properties of vector Mathieu-Gauss beam which is a fundamental solution of the vector wave equation in the paraxial limit. The general expression of this vector beam has been derived using the expansion of Mathieu beams on Bessel beams. It is found that the components of the vector Mathieu-Gauss field can be expressed as a combination of vector Bessel-Gauss beams of various order. The propagation equations of radial components of this vector beam have been established based on the Collins diffraction formula. The propagation properties of this vector beams for free space and through for a thin lens system have been analyzed with numerical calculations.

References