Paraxial propagation of Mathieu beams through an apertured $ABCD$ optical system

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Abstract

Based on the relationship between separable solutions of the Helmholtz equation, we expanded the field's amplitude associated with Mathieu beams in terms of Bessel beams. By using the expansion of the circ function into a finite sum of complex Gaussian functions, we derived an approximated analytical expression of the amplitude distribution of Mathieu beams at the output plane of any apertured paraxial $ABCD$ optical system. Some numerical calculations are performed and it is shown that our approach can substitute the usual numerical integration of the Helmholtz equation or the numerical calculation of the diffraction integral.

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1. Introduction

In recent years, the non-diffracting beams have received much attention in the optical domain. The interest in non-diffracting beams is due to the fact that, under ideal conditions, they propagate indefinitely without change in their transverse shape. This kind of beams has been used in different applications such as wireless communications, metrology, laser surgery, non-linear optics, optical tweezers, and so on.

The beams in question are exact separable solutions of the Helmholtz equation expressed in different coordinate systems [1–3]. Among them, we have Bessel [4,5], cosine [6], Mathieu [7–9] and Parabolic beams [10], which can be described as an adequate superposition of plane waves, whose vectors lie on a cone as it is...
imposed by the McCutchen theorem [11]. Recently, Gutiérrez-Vega et al. [7,8] have introduced and demonstrated experimentally the existence of Mathieu beams, based on the resolution of Helmholtz equation in the elliptic cylindrical coordinates. These beams are the elliptical generalization of the Bessel beams as it was signalized in [12]. More recently, based on the scalar diffraction theory applied to the experimental Durnin’s setup, Dartora et al. [13] obtained a closed analytical expression of the field associated with this class of beams. The advantage of this expression is that it leads to the relationship between Mathieu beams and Bessel beams expressed in cylindrical coordinates.

On the other hand, the technique of expanding the aperture function into a sum of complex Gaussian terms [14] have been used by many authors to derive approximate analytical propagation equations and factors of some beams propagating through an apertured paraxial optical systems [15–21]. The analytical results obtained with this technique meet a good agreement with the numerical calculations for the far field region; but a slight difference appears in the near field.

In the present work, we derive the analytical expansion in Bessel terms of all modes associated with the Mathieu beams by using the Whittaker integral and the angular spectrum decomposition [22]. The paraxial propagation equations of Mathieu beams through an apertured paraxial ABCD optical system are deduced from those of Bessel beams for which the treatment of expanding the circ function in terms of complex Gaussian terms is very well applied [15]. The paper is organized as follows: The analytical expressions of Mathieu beams in cylindrical coordinates as a function of Bessel beams are derived in Section 2. In Section 3, we will present the amplitude distribution of the Mathieu beams at the output plane of an apertured paraxial ABCD optical system. Some numerical calculations and analysis related to the propagation of this beams in free space, and through a thin lens and fractional Fourier transform will be given in Section 4. Finally, a conclusion is outlined in Section 5.

2. Field expressions of Mathieu beams in cylindrical coordinates

The representation in terms of plane waves of any non-diffracting beams can be given by the Whittaker integral [2,22],

\[ u(\rho, \varphi, z, t) = \exp(ik_z z - i\omega t) \int_0^{2\pi} A(\theta) \exp[i k \rho \cos(\theta - \varphi)] d\theta, \]

where \( \rho, \varphi, z \) are the cylindrical variables, \( \theta \) is the angular variable in the \( x-y \) plane, \( A(\theta) \) is the angular spectrum, and \( k_z = k \sin x_0 \), \( k_x = k \cos x_0 \) are the transverse and longitudinal components of the wave vector \( \vec{k} \). \( x_0 \) is the semi-angle of the cone on which lie the wave’s vectors associated with the plane waves. For Mathieu beams, the angular spectrum associated with the even and the odd modes are \( \text{ce}_m(\theta, q) \), \( \text{se}_m(\theta, q) \), respectively, where \( m \geq 0 \) for even modes and \( m \geq 1 \) for odd modes and \( q \) is the elliptic parameter given by \( q = h^2 k_z^2 / 4 \), where \( h \) is the interfocal separation [23]. As it is known, the functions \( \text{ce}_m \) and \( \text{se}_m \) can be expressed in term of Fourier series as:

\[
\begin{align*}
\text{ce}_{2n}(\eta, q) &= \sum_{j=0}^{\infty} B_{2j}^{(2n)}(q) \cos 2j \eta, \\
\text{ce}_{2n+1}(\eta, q) &= \sum_{j=0}^{\infty} A_{2j+1}^{(2n+1)}(q) \cos(2j + 1) \eta, \\
\text{se}_{2n}(\eta, q) &= \sum_{j=0}^{\infty} B_{2j}^{(2n+2)}(q) \sin(2j + 2) \eta, \\
\text{se}_{2n+1}(\eta, q) &= \sum_{j=0}^{\infty} A_{2j+1}^{(2n+1)}(q) \sin(2j + 1) \eta.
\end{align*}
\]
The coefficients $A_{2j}^{(2n)}(q)$, $A_{2j+1}^{(2n+1)}(q)$, $B_{2j+2}^{(2n+2)}(q)$, and $B_{2j+1}^{(2n+1)}(q)$ are characterized for each value of $q$ by a set of recurrence relations.

Replacing $A(\theta)$ by $\text{ce}_m(\theta, q)$ or $\text{se}_m(\theta, q)$, and taking into account the following equalities [24]:

$$\int_0^{2\pi} \exp\{i[-z\cos(\theta - \varphi) - n\theta]\} d\theta = 2\pi (-i)^n \exp(-im\varphi)J_n(z),$$  \hfill (3a)

$$J_n[z \exp(i\pi)] = \exp(in\pi)J_n(z),$$  \hfill (3b)

and

$$J_{-n}(z) = (-1)^n J_n(z),$$  \hfill (3c)

with $J$ is the Bessel function of the first kind, and $n$, $l$ are two integers, one can easily obtain the following field expressions associated with each mode of Mathieu beams in cylindrical coordinates:

\[
\begin{align*}
\text{uc}_{2n}(\rho, \varphi, q, z, t) &= 2\pi \exp(ikz - i\omega t) \sum_{j=0}^{\infty} A_{2j}^{(2n)}(q)(-1)^j \cos(2j\varphi)J_{2j}(k_1\rho), \\
\text{uc}_{2n+1}(\rho, \varphi, q, z, t) &= 2\pi \exp(ikz - i\omega t) \sum_{j=0}^{\infty} A_{2j+1}^{(2n+1)}(q)(i)^{2j+1} \cos[(2j + 1)\varphi]J_{2j+1}(k_1\rho), \\
\text{us}_{2n+2}(\rho, \varphi, q, z, t) &= 2\pi \exp(ikz - i\omega t) \sum_{j=0}^{\infty} B_{2j+2}^{(2n+2)}(q)(-1)^{j+1} \sin[(2j + 2)\varphi]J_{2j+2}(k_1\rho), \\
\text{us}_{2n+1}(\rho, \varphi, q, z, t) &= 2\pi \exp(ikz - i\omega t) \sum_{j=0}^{\infty} B_{2j+1}^{(2n+1)}(q)(i)^{2j+1} \sin[(2j + 1)\varphi]J_{2j+1}(k_1\rho).
\end{align*}
\]

(4)

From these expressions, we see that the Mathieu beams are an infinite sum of Bessel beams of various orders. Consequently, these relations will permit us to propagate analytically Mathieu beams using the well known integral’s diffraction and the propagation properties of Bessel beams.

3. Paraxial propagation of Mathieu beams

If we consider an apertured paraxial $ABCD$ optical system illuminated by one family of Mathieu beams of system (4), the propagation of this field through the considered system can be studied by using the Collins formula [25]. For the sake of simplicity, we consider only the first family of the above beams, i.e., the even modes $\text{uc}_{2n}$, the treatment of the other Mathieu family is the same. The relationship between the input and the output field distributions is given by

$$\text{uc}_{2n}^{(2)}(\rho_2, \varphi_2, z) = -\frac{ik}{2\pi B} \int_0^{\rho_0} \int_0^{2\pi} \rho_1 \text{uc}_{2n}^{(1)}(\rho_1, \varphi_1, z = 0) \times \exp \left[ ikz + \frac{k}{2B} (A\rho_1^2 - 2\rho_1\rho_2 \cos(\varphi_1 - \varphi_2) + D\rho_2^2) \right] d\rho_1 d\varphi_1,$$

where $\text{uc}_{2n}^{(1)}(\rho_1, \varphi_1, z = 0)$ is the field distribution at the point $(\rho_1, \varphi_1)$ in the input plane $z = 0$, and $\text{uc}_{2n}^{(2)}(\rho_2, \varphi_2, z)$ is field distribution at the point $(\rho_2, \varphi_2)$ in the output plane located at $z$ distance $A$, $B$, $C$ and $D$ are the elements of the transfer matrix characterizing the optical system. $k$ is the wave vector and $\rho_0$ is the radius of the input aperture. A constant phase term in Eq. (5) has been omitted which has non-influence on the output intensity distribution.

In the following, we use the non-dimensional variables $r_1 = \rho_1/\rho_0$, $r_2 = \rho_2/\rho_0$, $z = k\xi/2$ and $\xi = z/l_d$, where $l_d = k\rho_0^2/2$ is the diffraction length related to the aperture radius. To describe the finite size of the optical system, we introduce the aperture function:
As it is well known, this function can be expanded into a finite sum of complex Gaussian functions,

$$A_p(r_1) = \sum_{k=1}^{k=10} A_k \exp(-B_k r_1^2), \tag{7}$$

where $A_k$ and $B_k$ are the expansion and Gaussian coefficients, respectively, which could be found by a computer optimization [14]. By inserting this aperture function in the integral of Eq. (5) and by using the following result [23],

$$\int_{0}^{\infty} r_1 J_l(xr_1) J_l(\delta r_1) \exp(-v r_1^2) dr_1 = \frac{1}{2v} I_l(\frac{2\delta}{2v}) \exp\left(-\frac{x^2 + \delta^2}{4v}\right), \tag{8}$$

with $|\arg v| < \frac{\pi}{2}$, $\text{Re} x > 0$ and $\text{Re} \delta > 0$, where $I_l$ is the modified Bessel function of the first kind, and after some algebraic calculations, the output field distribution reads,

$$uc_{2n}^{(2)}(\rho_2, \varphi_2, z) = 2\pi \exp(i k z + i D \rho_2^2 / B') \sum_{k=1}^{k=10} A_k (1/A + i B_k B') \exp\left[-\left((z B')^2 + 4 \rho_2^2\right) / 4B'(B_k B' - i A)\right]$$

$$\times \sum_{j=0}^{\infty} A_{2j}^{(2)}(q)(-1)^j \cos(2j \varphi_2) J_{2j}(i z \rho_2 / B_k B' - i A), \tag{9}$$

with

$$B' = B / l_d. \tag{10}$$

This expression shows that the propagated beams preserve a similar expansion of Mathieu beams, with an added exponential term characterizing the paraxial approximation.

4. Numerical calculations and analysis

In this section, we perform some numerical calculations using Eq. (9) and we discuss the effect of the paraxial transformation by an $ABCD$ optical system on the truncated zeroth-order Mathieu beam for three particular cases: free space, thin lens and Fractional Fourier transform.

We note that in the preliminarily stage, we have used Matlab functions to calculate the eigenvalues of tridiagonal matrix associated with Mathieu equations and the Gutiérrez-Vega et al. [26] routines for the calculation of the expansion coefficients in the series of Eq. (4). As it is known the expansion coefficients vanish when the index $j$ tends to infinity, so less than twenty coefficients have been used in the following examples. The ratio between the coefficient $A_{40}^{(0)}$ and the coefficient that possesses the most significant weight $A_2^{(0)}$ is taken approximately equal to $10^{-25}$; with this ratio we can calculate the series with a good accuracy. For each value of $q$, we have chosen the number of expansion coefficients that gives the desired ratio. In the special case of $q = 0$, the characteristic value is reduced to zero and the expansion coefficients are reduced to $A_0^{(0)}$ and the others $A_{2j}^{(0)}$ vanish.

4.1. Free space propagation

The matrix transfer associated with a free space is
\( \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}. \)

In Fig. 1, we present the three-dimensional intensity pattern evolution of the zeroth-order Mathieu beam through the apertured free space. These plots were performed with the following parameters: \( q = 20, k_r = 33 \times 10^3 \text{ m}^{-1} \) and \( \rho_0 = 2 \text{ mm} \). It is seen that the obtained behaviors are in agreement with those performed numerically in [7,13].

![Fig. 1. Free space propagation of the truncated zeroth-order Mathieu beam for the parameters: \( \rho_0 = 2 \text{ mm}, q = 20 \) and \( k_r = 33 \times 10^3 \text{ m}^{-1} \): (a) propagation in \( x-z \) plane. (b) propagation in \( y-z \) plane.](image1)

![Fig. 2. Focalization of truncated zeroth-order Mathieu beam by a thin lens of focal length \( f = 0.6 \text{ m} \) for the parameters \( \rho_0 = 2 \text{ mm}, q = 20 \) and \( k_r = 15 \times 10^3 \text{ m}^{-1} \): (a) evolution in \( (x-z) \) plane. (b) evolution in \( (y-z) \) plane. (c) intensity distribution of truncated zeroth-order Mathieu beam in focal plane. (d) intensity distribution of truncated zeroth-order Bessel beam in focal plane.](image2)
4.2. Propagation through an apertured thin lens

This system is characterized by the following transfer matrix:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 - z/f & z \\
-1/f & 1
\end{pmatrix},
\]

(12)

here \(f\) is the focal length of the thin lens. To illustrate the propagation of Mathieu beam through this system, we present in Fig. 2(a) and (b) the evolution of the intensity distribution in the \((x - z)\) and \((y - z)\) planes. We give also in Fig. 2(c) the feature of the beam intensity in the focal plane. For comparison, the behavior of the Bessel beams in the focal plane is shown in Fig. 2(d). All these numerical calculations are performed with the same parameters \(q\) and \(\rho_0\), the others parameters are \(k_r = 15 \times 10^3\) m\(^{-1}\) and \(f = 0.6\) m. From these figures, we found the well known results on the focalization of the non-diffracting beams. Particularly, the existence of two maxima of the intensity, these maxima are located before and after the geometrical focal. The second peak of intensity behind the geometrical focal can be used to reconstruct

Fig. 3. Intensity distribution of truncated zeroth-order Mathieu beam in the output \((y-z)\) plane of FRT system for \(f = 0.75\) m, \(\rho_0 = 2\) mm, \(q = 20\), \(k_r = 33 \times 10^3\) m\(^{-1}\). (a) \(p = 0.1\); (b) \(p = 0.5\); (c) \(p = 0.7\); (d) \(p = 1\).
the beam with a desired profile [27,28]. Fig. 2(c) and (d) present the difference in the focal plane between the intensity distribution of zeroth-order Mathieu beam and zeroth-order Bessel beam. These plots show in both cases the angular spectrum associated with each beam, i.e., $A(\phi) = \text{cte}(\phi, q)$ for the Mathieu beam and $A(\phi) = \text{cte}$ for the Bessel beam.

4.3. Propagation through an apertured fractional Fourier transform

As it is known, the fractional Fourier transform (FRT), is an extension of the conventional Fourier transform. It was introduced into optics by Namias [29] as a mathematical tool to solve theoretical problems. There are at least two ways to implement optically the FRT: the first is based on the use of the graded-index medium, and the second uses the combination of lenses and space [30–32]. The $ABCD$ matrix associated with the $p$-order FRT is given by

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
\cos \varphi & f \sin \varphi \\
-\sin \varphi / f & \cos \varphi
\end{pmatrix},
$$

(13a)

with

$$
\varphi = p\pi/2,
$$

(13b)

and $f$ is the standard focal length. The conventional Fourier transform is obtained for $p = 4n + 1$, where $n$ is an integer.

By inserting the coefficients of the matrix Eq. (13a) in Eq. (9) one can obtain the intensity distribution of Mathieu beams in the output plane of FRT systems for different fractional order $p$. In Fig. 3, we present the dependence of the beam intensity on $p$, for the following parameters $f = 0.75$ m, $q$, $k$, and $\rho_0$ are kept the same as in the paragraph 4.1. In Fig. 4, we have presented the variation of the intensity in the interval $[-0.2 \text{ mm}, 0.2 \text{ mm}]$, even through the radius of the aperture is always conserved, i.e., $\rho_0 = 2 \text{ mm}$. From the Figs. 3 and 4, we deduce that the fractional order $p$ has a strong influence on the shape of the output distribution intensity of the beams when we consider the $(y - z)$ plane as a plane of propagation. However, we have a simple change in the width of the beams intensity for the $(x - z)$ plane.

Fig. 4. Intensity distribution of truncated zeroth-order Mathieu beam in the output $(x-z)$ plane of FRT system for $f = 0.75$ m, $\rho_0 = 2$ mm, $q = 20$, $k = 33 \times 10^3 \text{ m}^{-1}$ and $p \in \{0.1, 0.3, 0.5, 0.7\}$. 

5. Conclusion

Starting from the expansion of Mathieu beams in terms of plane waves, the analytical expression of the overlaps between the Mathieu beams and the Bessel beams was derived. Using Collins formula, we have derived the analytical expression of the field distribution of the truncated Mathieu beams in the output plane of an apertured \( ABCD \) paraxial system. Numerical calculations were performed to illustrate the paraxial propagation of these beams through free space, an apertured thin lens, and an apertured Fractional Fourier transform. It is shown that our results provide more convenient for treating the propagation and transformation of this kind of beams.

References