Focal shift in the axisymmetric Bessel-modulated Gaussian beam

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Abstract

The focal shift of the axisymmetric Bessel-modulated Gaussian beam with quadratic radial dependence (QBG beam) is investigated by using two different approaches. The first approach is based on the encircled-power criterion and the second one is the conventional method based on the axial irradiance. In this last case, a third-order equation governing the amount the relative focal shift is derived. Moreover, it is shown from numerical calculation results that the two approaches provide mainly the same results; the focal shift of the axisymmetric QBG beam depend on the Gaussian Fresnel number \( N_w \) and the beam parameter \( \mu \).

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1. Introduction

Recently, a novel class of beam expressed in cylindrical coordinate system, was introduced by Caron and Potvliege [1], namely, the Bessel-modulated Gaussian beams with quadratic radial dependence (QBG beam). It was shown that such class of beams has familiar collinear geometry of the Gaussian beam and also an interesting non-Gaussian features for certain values of its parameters. Particularly, it was demonstrated that the zeroth-order QBG beam, which is referred to as the axisymmetric QBG beam, can be expanded in Laguerre–Gauss modes and has a very flat axial profile when the beam parameter \( \mu \) is of order of unity.

On the other hand, the beam focusing and the associated focal shift have received a lot of
attention because of the theoretical and experimental interest. It is well known that when a light beam is focused by an aberration-free lens, the point of maximum irradiance of the diffracted field is not in general at the geometric focus but is shifted toward the lens. Such a phenomenon was established analytically by Li and Wolf [2,3] and it is currently known as focal shift. Recently, a lot of papers have studied the focal shift in different types of focused beams through a variety of systems [4–11]. It is shown that the magnitude of this shift depends strongly on the effective Fresnel number of the field. The shift is significant only when the Fresnel number is sufficiently small; for most imaging applications, it may safely be neglected. However, in laser systems the Fresnel number is of order of unity or smaller so the focal shift become relevant. As usual, the focal shift is determined by the axial maximum intensity. However for some types of beam, whose axial irradiance vanishes, the conventional method fails and an encircled-power criterion should be used [9].

In our previous paper, we have investigated the propagation properties of QBG beams through an unapertured optical paraxial ABCD system [12]. More recently, the beam propagation factor ($M^2$-factor) and the kurtosis parameter of such type of beams have been investigated [13,14]. However, in our knowledge, the case of the axisymmetric QBG beam has not been treated yet.

The present paper is aimed at studying the focusing of the axisymmetric QBG beam and the associated focal shift. In Section 2, the irradiance distribution of this beam through an aberration-free thin lens is established and focal shift is evaluated in two ways. In a first approach, the focal shift is evaluated based on the encircled-power criterion. In the second approach, we use the conventional treatment based on the axial maximum irradiance. In Section 3, Numerical calculation results are presented to illustrate the existence of the focal shift and its variation against the Fresnel number $N_w$ and beam parameter $\mu$. A comparison between numerical results of the two approaches is given. In conclusion, the main results of this study are summarized.

2. Irradiance distribution of the focused axisymmetric QBG beam

In the cylindrical coordinate system ($r,\phi,z$) the field distribution $E(r,\phi,z)$ of the axisymmetric QBG beam at the plane $z=0$ reads [1]

$$E(r,\phi,z=0) = J_0\left(\frac{\mu^2}{w_0^2}\right) \cdot \exp\left(\frac{r^2}{w_0^2}\right),$$

(1)

where $J_0$ denotes the Bessel function of order zero, $w_0$ is the waist width of the Gaussian beam, $\mu$ is a beam parameter which is complex-valued in general.

The propagation equation of the axisymmetric QBG beam passing through an ideal thin lens of focal length $f$ takes the form [12]

$$E(r,\phi,z) = \frac{w_0}{w(z)} J_0\left(\frac{\mu^2}{w^2(z)}\right) \times \exp\left\{\left(\frac{1-i\omega f z}{w^2(z)} + \frac{i k D}{2 R}\right) r^2\right\} \exp(iz),$$

(2a)

where $k$ is the wave number, $z_R = k w_0^2/2$ denotes the Rayleigh range of the Gaussian mode, $A, B, D$ are the elements of the $ABCD$ lens transfer matrix at any output plane $z$. This matrix is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-z/f & z \\ -1/f & 1 \end{pmatrix},$$

(2b)

and the waist width is

$$w(z) = w_0 \sqrt{A^2 - (\mu^2 + 1) \frac{B^2}{z_R} + 2 i \frac{AB}{z_R}}.$$  

(2c)

The irradiance distribution $I(r,\phi,z)$ of the focused axisymmetric QBG beam at the output plane $z$ is defined as the square modulus of the field, i.e.

$$I(r,\phi,z) = |E(r,\phi,z)|^2.$$  

(3)

Substituting Eq. (2a) into Eq. (3), and after some algebraic manipulations the irradiance distribution can be expressed as
\[ I = \frac{1}{\sqrt{\beta^2 + \gamma^2}} \exp \left[ \frac{2(\beta - 2d^2)}{w_0^2(\beta + \gamma^2)} r^2 \right] \times J_0 \left( \frac{\mu^2}{w_0^2(\beta + i\gamma)} \right) J_0 \left( \frac{\mu^2}{w_0^2(\beta - i\gamma)} \right), \] (4a)

where

\[ \beta = A^2 - (\mu^2 + 1) \frac{B^2}{z_R} \] (4b)

and

\[ \gamma = \frac{2AB}{z_R}. \] (4c)

It must be noted that in Eqs. (4a) and (4b), and in the following, we limit ourselves to real values of \( \mu \). From Eq. (4a), it follows that the irradiance distribution depends, in addition to the spatial coordinates \( r \) and \( z \), on the parameters of the beam, \( w_0 \) and \( \mu \), and the focal length \( f \) of the lens.

2.1. Focal shift based on the encircled-power criterion

The focusing properties of a beam can be investigated theoretically, in a general way, by using a convenient definition of the beam width. We use the encircled-power criterion, as was made by Green and Hall [9] focused focused annular vector Bessel Gaussian beams. In this approach, the width \( W \) of the beam is defined as the radius within which 80% of the beam’s power is enclosed.

The power carried may be found by integrating the irradiance distribution over the transverse plane. So, for the cylindrical coordinate system, the beam width is given by

\[ \int_0^{2\pi} \int_0^W I(r, \phi, z) r \, dr \, d\phi \int_0^{2\pi} \int_0^\infty I(r, \phi, z) r \, dr \, d\phi = 0.8. \] (5a)

For the rotationally symmetrical (axisymmetric) QBG beam Eq. (5a) can be rewritten with simple integrals, as

\[ \int_0^W I(r, \phi, z) r \, dr \int_0^\infty I(r, \phi, z) r \, dr = 0.8. \] (5b)

The focal plane is then defined as the plane \( z_0 \) in which \( W \) reaches its minimum value. From Eq. (5b) the smallest width \( W_0 \) and its position, namely the position of the focal plane \( z_0 \), can be performed numerically. As result, the relative focal shift defined by \( \Delta z = (z_0 - f)/f \) is found.

2.2. Focal shift based on the axial maximum irradiance

In the conventional treatment, the location of the focal plane is defined as the point \( z_{\text{max}} \) of axial maximum irradiance. From Eq. (4a), the irradiance distribution of the focused axisymmetric QBG beam on the z-axis reduces to

\[ I(r = 0, \phi, z) = \frac{1}{\sqrt{\beta^2 + \gamma^2}}. \] (6)

So, to get the point \( z_{\text{max}} \) of axial maximum irradiance one may write

\[ \frac{dI(r = 0, \phi, z)}{dz} = 0. \] (7a)

This last equation can be rewritten as

\[ \frac{dI(r = 0, \phi, z)}{d(\Delta z)} \cdot \frac{d(\Delta z)}{dz} = 0, \] (7b)

where \( \Delta z \) represents the relative focal shift of the focused axisymmetric QBG beam, and

\[ \Delta z = \frac{z_{\text{max}} - f}{f}. \] (7c)

On substituting from Eq. (6) into Eq. (7b), and after straightforward calculations one obtains the following third-order equation

\[ \Delta z^3 + a_1 \Delta z^2 + a_2 \Delta z + a_3 = 0, \] (8a)

where the coefficients \( a_1, a_2, a_3 \) are given by

\[ a_1 = \frac{3h(1 - h^2g^2) - 6}{a}, \] (8b)

\[ a_2 = \frac{-3h^2g^2 + h - 2}{a}, \] (8c)

and

\[ a_3 = \frac{-h^2g^2}{a}. \] (8d)
with
\[ a = -\left(1 - h \cdot g^2\right)^2 - 4, \]  
(8e)
\[ h = \mu^2 + 1 \]  
(8f)
and
\[ g = \frac{1}{\pi N_w}. \]  
(8g)

In Eq. (8g), \( N_w = \frac{w_0^2}{2z} \) is the Fresnel number associated with the Gaussian part.

The resolution of Eq. (8a) [15] may directly provide the relative focal shift for any arbitrary values of the beam parameters. Among the three roots of the third-order equation, the convenient one must be real valued and satisfies the condition,
\[ -1 < \Delta z < 0. \]  
(9)

For the limiting case of \( \mu = 0 \), the axisymmetric QBG beam reduces to the pure Gaussian beam, and Eq. (8a) leads the following solution
\[ \Delta z_G = -\frac{1}{1 + (\pi N_w)^2}. \]  
(10)

which is in agreement with well known expression for the Gaussian beam, given earlier by Li and Wolf in [3].

3. Numerical results

Numerical calculations were performed to illustrate the focusing in axisymmetric QBG beam and to study the relative focal shift against the Fresnel number \( N_w \) and the parameter \( \mu \). In Fig. 1, a plot of the normalized beam width, \( \frac{W(z)}{w_0} \), is presented against the relative distance \( z/f \) for some values of \( N_w \), and for \( \mu = 0.5 \). It is seen that the normalized beam width reaches its minimum value before the geometric focal length, i.e., the real focal plane is shifted toward the lens. This focal plane moves toward the geometric focus when the Fresnel number increases for a fixed value of \( \mu \). Similarly, Fig. 2 plots the normalized beam width against \( z/f \) for several values of \( \mu \), and for \( N_w = 5 \). It is shown that the focal plane moves toward the lens when \( \mu \) increases. We note that for \( \mu = 0 \), on obtains the well known width profile of the focused pure Gaussian beam along \( z \)-axis.

These results are further confirmed by plots of Fig. 3 in which the relative focal shift \( \Delta z \) is depicted versus the Fresnel number \( N_w \), for several values of \( \mu \). It is clearly seen that the modulus of relative focal shift \( |\Delta z| \) increases with decreasing \( N_w \) and/or increasing \( \mu \).

We have, on the other hand, performed numerical calculations of the focal shift from Eq. (8a), and the curves of variation of \( \Delta z \) with \( N_w \) are
presented in Fig. 4. For comparison, we have presented in this figure the corresponding results which are obtained from the encircled-power criterion. It is clearly seen that the curves corresponding to both approaches are mainly identical. The quantitative difference for both cases, in fact, is of order of 0.001. The dependence of \( D_z \) on the parameter \( l \) is depicted in Fig. 5, from which we see that \( |D_z| \) increases with increasing \( l \) if \( N_w \) is kept fixed.

4. Conclusion

A detailed study of focusing properties and focal shift of the axisymmetric QBG beam passing through a non-apertured aberration-free lens has been made. Two approaches have been used: the encircled-power criterion and the axial maximum irradiance. Numerical calculation results based on the encircled-power criterion, show that the position of the smallest width \( W_0 \) of the focused axisymmetric QBG beam (focal plane position) moves toward the lens with decreasing the Fresnel number \( N_w \) (for a fixed \( l \)) or increasing \( l \) (for a fixed \( N_w \)). On the other hand, the method of the maximum axial irradiance leads to a third-order equation from which the relative focal shift is performed directly. The results based upon the two different approaches are mainly identical; the modulus of relative focal shift increases with decreasing \( N_w \) and increasing \( l \). Although the encircled-power criterion provides a more general definition of the focal plane, it involves two integrals of Eq. (5a) which are difficult to perform analytically, and are in general calculated numerically. So, for beams whose axial intensity does not vanish, the value of the focal shift should be in principle an approximation to that obtained from the maximum axial irradiance method.

Finally, we would like to mention that in this work, only the ideal thin lens has been considered.
A further study of the focusing properties taking into account the spherical aberration of the lens, specially the improvement of the focusing, and the case of the apertured QBG beams are expected.

References