Focusing properties and focal shift in hyperbolic-cosine-Gaussian beams

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Received 22 February 2005; received in revised form 26 April 2005; accepted 26 April 2005

Abstract

Based on the generalized Huygens–Fresnel diffraction integral the expression of the focused hyperbolic-cosine-Gaussian (ChG) beams is derived for any transverse plane in the focal region. The properties of the focused ChG beams and the focal shift are studied, and their dependence on the Fresnel number and the decentered parameter \( \beta \) are illustrated with detailed numerical calculations.

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Keywords: Hyperbolic-cosine-Gaussian beams; Focal shift; Huygens–Fresnel diffraction integral

1. Introduction

In the end of the last century, various closed form paraxial solutions of the Helmholtz equation have been the subject of several works. A particularly interesting class of beams, a so-called Hermite-sinusoidal-Gaussian (HSG) beams, was introduced in the rectangular coordinate system by Casperson and co-workers [1–3]. Among the sets of HSG beams, the hyperbolic-cosine-Gaussian (ChG) beams may have many interesting applications because their profile can resemble closely the flat-topped field distributions by an appropriate choice of the decentered parameter \( \beta \).

Recently, the practical generation of these beams by using graded-phase mirrors has been discussed in detail by Yu et al. [4]. In our last papers [5–7], we have studied the parametric characterization of this class of beams including the beam propagation factor (\( M^2 \)-factor) and the kurtosis parameter \( k \) in the
two-dimensional space. In the present paper, the focusing properties of ChG beams are examined in the three-dimensional case by using the generalized Huygens–Fresnel diffraction integral. The focal shift [8,9] is studied based on the axial irradiance distribution, and its dependence on the Gaussian Fresnel number and the decentered parameter $\beta$ is described with detailed numerical illustrations.

2. Field distribution of the focused ChG beams

Consider a three-dimensional ChG beam, in the cartesian coordinate system, whose field distribution $E(x_0, y_0, z = 0)$ at the $z = 0$ plane reads [1–3]

$$E(x_0, y_0, z = 0) = \cosh(\Omega x_0) \cosh(\Omega y_0) \exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right),$$

where $w_0$ is the waist of the Gaussian amplitude distribution, $\Omega$ is the normalized parameter of the ChG beam, and $\cosh$ denotes the hyperbolic-cosine function. Eq. (1) can be rewritten as

$$E(u_0, v_0, z = 0) = \cosh(\beta u_0) \cosh(\beta v_0) \exp[-(u_0^2 + v_0^2)],$$

where $\beta = w_0 \Omega$ is the decentered parameter and, $u_0 = \frac{x_0}{w_0}$ and $v_0 = \frac{y_0}{w_0}$ are the normalized transverse coordinates. A preliminary illustration of ChG beam at its initial ($z = 0$) plane can be found in Fig. 1, where the isophotes (contours lines) of the beam are presented for some values of $\beta$. It is clearly seen that for a small value of $\beta$ ($0 \leq \beta \leq 1.5$), the profile of the beam consists of a central lobe with the peak intensity on the $Z$-axis. The contour lines which are concentric circles for Gaussian beam ($\beta = 0$) become square-shaped with increasing $\beta$ ($\beta = 1.5$). Whereas $\beta$ is larger than 2, the ChG profile exhibits four lobes and the peak intensity is no longer achieved on the $Z$-axis. Furthermore, these lobes move away from the $Z$-axis when $\beta$ is increased.

The propagation of the ChG beam through an unapertured converging thin lens of focal length $f$, placed in the plane $z = 0$ (see Fig. 2) is described by the Huygens–Fresnel diffraction integral taking the form [10,11]

$$E(x, y, z) = \left(\frac{ik}{2\pi B}\right) \int \int E(x_0, y_0, z = 0) \exp\left\{\frac{-ik}{2B} \left[A(x_0^2 + y_0^2) - 2(ax_0 + by_0) + D(x^2 + y^2)\right]\right\} \, dx_0 \, dy_0,$$

where $k$ is the wave number, $A$, $B$ and $D$ are elements of the transfer matrix lens system which reads as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - z/f & z \\ -1/f & 1 \end{pmatrix}.$$ (4)

For convenience, an unimportant phase factor depending on the axial distance $z$ is omitted in Eq. (3). The substitution from Eq. (1) into Eq. (3) yields

$$E(u, v, z) = \left(\frac{ik}{2\pi B}\right) T(u)T(v),$$

where

$$T(z) = w_0 \exp\left(-\frac{ikw_0^2}{2B} z^2\right) \int \cosh(\beta u_0) \exp(-u_0^2) \exp\left[-\frac{ikw_0^2}{2B} (4u_0^2 - 2axu_0)\right] \, du_0$$

with $z$ sets for a normalized transverse coordinate $u$ or $v$ in the $z$ plane. After some algebraic calculations, the integral expression $T(z)$ reads as
\[ T_z = \sqrt{\pi} \frac{w_0}{p} \exp(\beta/2p)^2 \exp \left\{ -\frac{ikw_0^2}{2B} x^2 \left( 1 - \frac{ik}{2B} \left( \frac{w_0}{p} \right)^2 \right) \right\} \cosh \left( \frac{ikw_0^2}{2B} \frac{\beta}{p^2} x \right) \]

with
\[ p = \sqrt{1 + \frac{ikA}{2B} w_0^2}. \]
By introducing the Fresnel number associated with the Gaussian term, \( N_w = \frac{\pi^2}{\lambda f} \), the amplitude expression of Eq. (5a) can be rewritten as

\[
E(u, v, z) = \frac{\pi}{p^2} \exp \left( \frac{\beta}{2p} \frac{z}{z/f} \right) \exp \left( \frac{Q(u^2 + v^2)}{z/f} \right) \cosh(Su) \cosh(Sv),
\]

where

\[
Q = \frac{i\pi N_w}{z/f} \left( \frac{i\pi N_w}{z/f} \frac{1}{p^2} - 1 \right)
\]

and

\[
S = \frac{i\pi N_w \beta}{z/f} p^2.
\]

For the special case of \( \beta = 0 \), Eq. (7a) reduces to

\[
E(u, v, z) = \frac{i\pi}{p^2} \frac{N_w}{z/f} \exp \left( \frac{Q(u^2 + v^2)}{z/f} \right),
\]

which is the well-known propagation expression of the pure Gaussian beam [11].

We know that the irradiance distribution (intensity) of the focused ChG beam is defined as

\[
I(u, v, z) = |E(u, v, z)|^2.
\]

The substitution from Eq. (7a) into Eq. (9), and after some algebraic calculations yields

\[
I(u, v, z) = I_a(z) \exp \left( \frac{-2(\pi N_w/s)^2(u^2 + v^2)}{1 + [(\pi N_w/s)(s - 1)]^2} \right) \cosh(2S_1 u) \cosh(2S_2 v) \cosh(2S_1 v) \cosh(2S_2 u),
\]

where \( s = \frac{z}{f} \) is the relative axial distance,

\[
I_a(z) = \frac{\pi}{(s/N_w)^2 + [\pi(s - 1)]^2} \exp \left( \frac{\beta^2}{1 + [\pi N_w/(s - 1)]^2} \right)
\]

and, \( S_1 \) and \( S_2 \) are the real and imaginary parts of \( S \) which are given respectively by

\[
S_1 = \beta \frac{(\pi N_w)^2(s - 1)}{s^2 + (\pi N_w)^2(s - 1)^2}
\]

and

\[
S_2 = \beta \frac{\pi N_w/s}{1 + [\pi N_w/(s - 1)]^2}.
\]

The irradiance distribution along the \( Z \)-axis is obtained by substituting \((u, v) = (0, 0)\) into Eq. (10a), that is we get

\[
I(0, 0, z) = I_a(z).
\]

It is readily seen from this last equation that the axial intensity of the focused ChG beam would never vanish. Consequently, the position \( z_m \) of the maximum axial irradiance, namely the position of the focal plane, should be determined from the condition
On substituting from Eq. (11) into Eq. (12), and carrying out the somewhat lengthy algebra we find that

\[ (g + 1)^3 s^3 + g[\beta^2 - 3(g + 1)]s^2 + g[(1 + 3g) - \beta^2]s - g^2 = 0 \]

with \( g = \pi^2 N^2_w \). The relative focal shift which is defined by

\[ \Delta z = s - 1 \]

must then be a root of the equation

\[ (g + 1)^2 \Delta z^3 + [g(\beta^2 + 3) + 3]\Delta z^2 + [g(\beta^2 + 1) + 3]\Delta z + 1 = 0. \]

This last equation indicates that the amount of the relative focal shift \( \Delta z \) depends on the decentered parameter \( \beta \) and the Fresnel number \( N^2_w \). For the limiting case of \( \beta = 0 \) the ChG beam reduces to the pure Gaussian beam, and in this case Eq. (15) has the following root:

\[ 0.8, 0.9, 1.0, 1.1, 1.2 \]

\[ 0.0, 0.2, 0.4, 0.6, 0.8, 1.0 \]

\[ N^2_w = 1, 5, 20 \]

\[ z/f \]

**Fig. 3.** Normalized axial intensity of the focused ChG beam for \( \beta = 0, 1, 2 \) and 4, and for \( N^2_w = 1, 5 \) and 20.
\[ \Delta z^G = \frac{-1}{1 + g}, \]

which is the well-known focal shift expression for the Gaussian beam [9].

The cubic equation (15) may have three roots, among them the convenient solution must be real valued and satisfies the condition
\[ -1 < \Delta z < 0. \]

Numerical resolution of Eq. (15) [12] will give directly the relative focal shift for any values of \( N_w \) and \( \beta \).

3. Numerical results and analysis

Numerical calculations were performed using Eqs. (10a), (11) and (15) to illustrate the focusing properties and the focal shift of the ChG beams. Fig. 3 presents the normalized axial intensity of the focused ChG beam for various decentered parameter \( \beta \) (\( \beta = 0, 1.5, 2 \) and 4) and Fresnel number \( N_w \) (\( N_w = 1, 5 \) and 20). From these plots, it can be seen that the peak of the axial intensity is reached somewhat before the point of geometrical focus. This means that the location of the real focal plane \( z_m \) is shifted slightly...
toward the lens. Furthermore, it is clearly shown that the focal plane moves toward the geometrical focus when $\beta$ or $N_w$ are decreased. The structure of the ChG beam at the focal plane $z_m$ is illustrated in Fig. 4, which shows the isophotes of the beam for different values of $\beta$ and for $N_w = 1$. It is readily seen that the focusing leads to the change of the beam profile; one can note the apparition of a central lobe with respect to plots of Fig. 1. It means that the ChG beams cannot retain the shapes unchanged upon propagating. This result can be deduced directly from examining the irradiance expression of Eq. (10a); one can note the presence of new terms of the form cosine-Gaussian functions. The physical explanation is that the cosh-Gaussian beams belong to the class of HSG beams with complex argument [1].

The dependence of the relative focal shift $\Delta z$ on the Fresnel number $N_w$ is depicted in Fig. 5(a), from which it is seen that $|\Delta z|$ vanishes asymptotically when $N_w$ is greater than say six whereas it increases with decreasing values of $N_w$ (when $\beta$ is kept fixed), $|\Delta z|$ becomes important when small values of $N_w$ are used. From the same plots it is seen that $|\Delta z|$ increases with decreasing $\beta$ (when $N_w$ is kept fixed) and this is confirmed by plots of Fig. 5(b).

4. Conclusion

A detailed study of focusing properties and focal shift of hyperbolic-cosine-Gaussian (ChG) beams through an unapertured thin lens have been made using the generalized Huygens–Fresnel diffraction integral. The propagation expression of the ChG beam at any transverse plane in the focal region has been derived, and isophotes of the beam have been depicted to show the focusing of the beam. A cubic equation determining the relative focal shift has been established based on the axial maximum intensity. Numerical calculations have shown that the position of real focal plane, is not coincident with the geometrical focus but is somewhat shifted toward the lens. Numerical calculations have shown that the modulus of relative focal shift increases with decreasing $N_w$ or the decentered parameter $\beta$.

References