KURTOSIS FACTOR
OF SOME TRUNCATED AND NON-TRUNCATED LASER BEAMS

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Abstract
Based on the intensity moments formalism, the closed-form expressions of the kurtosis factor of some truncated and non-truncated laser beams with rectangular symmetry, as Hermite-cosh-Gaussian and super-Gaussian beams, are derived and illustrated numerically. Some particular cases are also deduced. Our results are consistent with previous works and constitute a more generalized study of kurtosis factor of laser beams.

Keywords : Kurtosis factor; Super-Gaussian; Hermite-cosh-Gaussian; Cosh-Gaussian; Hermite-Gaussian; Hard-edged aperture.

1. Introduction
In recent papers, an interesting class of solutions of the paraxial wave equation in the rectangular coordinate system, known as Hermite-sinusoidal-Gaussian (SHG) modes, has been introduced by Casperson and Tovar [1,2]. It is shown that these modes represent the more general rectangular-symmetrical beams and the Gaussian, Hermite-Gaussian and sinusoidal-Gaussian beams can be regarded as their particular cases. On the other hand, the super-Gaussian beams represent an area of interest because of their flat-topped intensity profiles [3].

For defining the propagation and the transformation of hard-edge diffracted laser beams, one needs to know the energy distribution at the artificial waist. So, a lot of research, based on the intensity moments formalism, have been performed to study the parametric characterization of some paraxial solutions of the Helmholtz equation [4-10].

In previous papers, we have discussed a detailed study of propagation properties [9] and investigated a parametric characterization of hard-edge diffracted Hermite-cosh-Gaussian beams (which refer to as HChG) [10]. The closed-form expression of propagation factor $M_{2G}$ of this family of beams has been established by using the generalized second-order intensity moments [5]. Others tool parameters can be used to reveal informations about the spatial characteristics of the beams, like the skewness and kurtosis factors.

The coefficient of skewness, which corresponds to the third-order intensity moment, vanishes for the above beams because their profiles are symmetric about their means. The kurtosis factor $k$, which is expressed in terms of second and fourth intensity moments can be used to distinguish intensity distributions which are tall and sharp from those that are short and wide [11]. Recently, Saghafi et al [12] have used the kurtosis factor, as a second characterizing parameter in order to analyze the standard and elegant-Hermite-Gaussian beams. As far as we know, the kurtosis factor of the more generalized HChG modes and the super-Gaussian beams have not been studied elsewhere.

This paper is devoted to studying the kurtosis factor as a characterizing parameter for some laser beams in both truncated and non-truncated cases. Starting from the definition of the kurtosis factor which is given in Section 2, the closed-form expression of kurtosis factor of Hermite-cosh-Gaussian beams has been derived in Section 3. Some particular cases are deduced, like Gaussian, cosh-Gaussian and Hermite-Gaussian beams, in Section 4. In Section 5, the analytical formula of kurtosis factor of super-Gaussian beams is investigated. Some numerical examples are illustrated and discussed in Section 6.

2. Kurtosis factor expression
The kurtosis factor of a laser beam, in one dimension, is defined as [11,13]

$$k = \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2}, \quad (1)$$

where $\langle x^2 \rangle$ and $\langle x^4 \rangle$ are irradiance second and fourth-order moments in the spatial domain, respectively.

Assume that a hard-edge aperture with half width $a$ is positioned in the waist plane $z=0$ where the amplitude of the field is given by $E(x,0)$. After normalizing, we can express the kurtosis factor as...
\[k = \int_a^b x^4 |E(x,0)|^2 \frac{dx}{(\int_a^b x^3 |E(x,0)|^2 \, dx)^2} \int_a^b |E(x,0)|^2 \, dx. \tag{2}\]

The integrals in Eq. (2) will be expressed in terms of the aperture width and the parameters of the beams. On the other hand, the corresponding expression of the non-truncated case will be obtained when a tends to infinity.

### 3. Kurtosis factor of truncated HChG beams

The field distribution of this beams family at the \(z = 0\) plane is given by [1]

\[E_z(x,0) = A_0 H_n(\sqrt{\frac{x}{\omega_0}}) \cosh(\Omega x) \exp\left(-\frac{x^2}{\omega_0}\right), \tag{3}\]

where \(m\) is the mode index associated to the Hermite polynomial function, \(\omega_0\) is the waist width of the Gaussian amplitude distribution, \(\Omega\), is the parameter associated with the \(\cosh\) function and \(A_0\) is the amplitude at the central position of \(x = z = 0\). In the following, \(A_0\) is assumed to be equal to unity for the sake of simplicity.

Using the expanding form of the Hermite polynomial and the probability integral of the \(x^2\) distribution defined by [14]

\[P(x^2/\nu) = \frac{1}{2^{\nu/2} \Gamma(\nu/2)} \int_0^{\infty} e^{-x^2/\nu} \, dx, \tag{4a}\]

which is related to the incomplete gamma function \(\gamma\) and the Euler function \(\Gamma\) by [14]

\[P(x^2/\nu) = \frac{\gamma(x^2/\nu, \nu)}{\Gamma(\nu/2)}, \tag{4b}\]

and after some algebraic manipulations, which were exposed in detail in Ref. [10], one obtains the following expressions of integrals of Eq. (2)

\[
\int_a^b |E(x,0)|^2 \, dx = \frac{\omega_0}{2} 4^{m}(\nu t)^2
\]

\[
\sum_{n=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \sum_{j=0}^{[m/2]} A_{j,m} \left[ e^{\beta^2/2} S_{1,m+1} + S_{2,m+1} \right], \tag{5a}\]

\[
\int_a^b x^2 |E(x,0)|^2 \, dx = \frac{1}{2} \sum_{n=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \sum_{j=0}^{[m/2]} A_{j,m} \left[ e^{\beta^2/2} S_{1,m+1} + S_{2,m+1} \right], \tag{5b}\]

\[
\sum_{m=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \sum_{j=0}^{[m/2]} A_{j,m} \left[ e^{\beta^2/2} S_{1,m+2} + S_{2,m+2} \right]. \tag{5c}\]

The coefficients appear in these equations are expressed as

\[A_{j,m} = \frac{(-1)^j}{j!(m - 2j)! 4^j}, \tag{6a}\]

\[A_j = \frac{\omega_0}{2} 4^{m}(\nu t)^2 \sum_{n=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \sum_{j=0}^{[m/2]} A_{j,m} \left[ e^{\beta^2/2} S_{1,m+1} + S_{2,m+1} \right], \tag{6b}\]

\[\sum_{m=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \sum_{j=0}^{[m/2]} A_{j,m} \left[ e^{\beta^2/2} S_{1,m+2} + S_{2,m+2} \right]. \tag{6c}\]

\[S_{1,m} = \frac{2(m - n - j)}{2m(m - n - j)} \sum_{k=0}^{[m/2]} \frac{\Gamma(k + 1)}{k!(2m - n - j - k)!} t^{k-2} (\int(-1)^k \left[ 4\delta^2 + \beta^2 \right]^2 (k + 1) + P(4\delta^2/k + 1) \right), \tag{6d}\]

\[S_{2,m} = \gamma \left( m - n + j + \frac{1}{2} \right) \left[ 4\delta^2/(2m - n - j + 1) \right], \tag{6e}\]

where \(\delta = \alpha - \beta/2\) is the truncation parameter,

\[\beta = \Omega_0 \omega_0\) is the modal parameter and \(\alpha = \frac{a}{\omega_0}\).

The quantity \([m/2]\) is the integer part of \(m/2\).

Note that if \(\delta < 0\), the second term of Eq. (6c) into the bracket may be multiplied by \((-1)^{k-1}\). Finally, the kurtosis factor can be written as

\[k = \frac{\lambda A B C}{C^2}, \tag{7}\]

where

\[A = \sum_{m=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \left[ e^{\beta^2/2} S_{1,m+1} + S_{2,m+1} \right], \tag{8a}\]

\[B = \sum_{m=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \left[ e^{\beta^2/2} S_{1,m+1} + S_{2,m+1} \right], \tag{8b}\]

and

\[C = \sum_{m=0}^{[m/2]} \sum_{n=0}^{[m/2]} A_{n,m} \left[ e^{\beta^2/2} S_{1,m+1} + S_{2,m+1} \right]. \tag{8c}\]

Eq. (7) is a closed–form of the kurtosis factor of HChG beams passing through a hard-edged aperture. This formula can be considered as a general expression from which we will deduce, in the following, the kurtosis factor of some laser beams. The corresponding analytical expression for the non-truncated case is obtained by letting \(\alpha \to \infty\) in Eq. (7).

### 4. Special cases

#### 4.1 Hermite-Gaussian beams

If we assume \(\beta=0\), the HChG beam reduces to the Hermite-Gaussian one. For this case, it can be shown immediately that Eq. (6c) is similar to Eq. (6d), i.e. \(S_{1,m} = S_{2,m}\), then the analytical expression of the kurtosis factor is given by

\[k_{\text{HG}} = \frac{\lambda A_{\text{HG}} B_{\text{HG}}}{C_{\text{HG}}^2}, \tag{9}\]

where
The non-truncated case is obtained when the aperture width approaches infinite.

### 4.2 Truncated Gaussian beams

The kurtosis factor of a truncated Gaussian beam is obtained by assuming both \( \beta \) and \( m \) equal to zero in the Eq. (7). So, it can be expressed as

\[
\kurtosity_G = \frac{A_G B_G}{C_G},
\]

where

\[
A_G = \frac{3\sqrt{\pi}}{4} P(4\alpha^2 / 5),
\]

\[
B_G = \sqrt{\pi} P(4\alpha^2 / 1),
\]

and

\[
C_G = \frac{\sqrt{\pi}}{2} P(4\alpha^2 / 3).
\]

As it is well-known, the kurtosis factor for non-truncated case is equal to 3. This particularity is obtained by the property of the probability integral [14]

\[
\lim_{x \to \pm \infty} P(x^2 / \nu) = 1.
\]

On the other hand, the expression of \( \kurtosity_G \) can be written in terms of the error function. So, we have

\[
k = \operatorname{erf}(\sqrt{2}a) \times \frac{[3\pi \operatorname{erf}(\sqrt{2}a) - 4\sqrt{2}\pi \alpha(2\alpha^2 + 3/2)e^{-2a^2}]}{[\sqrt{2}\pi \operatorname{erf}(\sqrt{2}a) - 2\sqrt{2}2ae^{-2a^2}]^2},
\]

To get this last equation, we have used Eqs. (26.4.19), (6.5.16) and (26.4.8) of Ref. [14].

### 4.3 Truncated cosh-Gaussian beams

If the mode index \( m \) is equal to zero, the HChG beam reduces to the cosh-Gaussian one. In this case the kurtosis factor is given by

\[
\kurtosity_{CG} = \frac{A_{CG} B_{CG}}{C_{CG}},
\]

where

\[
A_{CG} = e^{2} S_{1,1} + S_{2,1},
\]

\[
B_{CG} = e^{2} S_{1,0} + S_{2,0},
\]

and

\[
C_{CG} = e^{2} S_{1,1} + S_{2,1},
\]

with

\[
S_{1,0} = \frac{\sqrt{\pi}}{2} [P(4(\delta + \beta)^2 / 1) + P(4\delta^2 / 1)],
\]

\[
S_{2,0} = \sqrt{\pi} P(4\alpha^2 / 1),
\]

\[
S_{1,1} = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{(\beta / \sqrt{2})^{2-k} k!}{(2-k)!} \left[ (-1)^k P(4(\delta + \beta)^2 / k + 1) + P(4\delta^2 / k + 1) \right],
\]

\[
S_{2,1} = \frac{3\sqrt{\pi}}{4} P(4\alpha^2 / 5),
\]

\[
S_{1,2} = \frac{12}{\sqrt{2}} [P(4(\delta + \beta)^2 / k + 1) + P(4\delta^2 / k + 1)],
\]

\[
S_{2,2} = \frac{\sqrt{\pi}}{2} P(4\alpha^2 / 3),
\]

Using again Eqs. (26.4.19), (6.5.16) and (26.4.8) of Ref. [14] and after tedious calculations, the last terms are expressed as

\[
S_{1,0} = \frac{\sqrt{\pi}}{2} \{\operatorname{erf}(a_{-}) - \operatorname{erf}(a_{+})\},
\]

\[
S_{2,0} = \frac{\sqrt{\pi}}{2} \{\operatorname{erf}(\sqrt{2}a)\},
\]

\[
S_{1,1} = \frac{\sqrt{\pi}}{4} \{1 + a^2\} [\operatorname{erf}(a_{-}) - \operatorname{erf}(a_{+})]
\]

\[
+ \frac{1}{2} \left[ a_{-}\exp(-a_{+}^2) - a_{+}\exp(-a_{-}^2) \right],
\]

\[
S_{2,2} = \frac{3\sqrt{\pi}}{4} \operatorname{erf}(\sqrt{2}a) - \left( \frac{3 + 4\alpha^2}{\sqrt{2}} \right) a e^{-a^2},
\]

\[
S_{1,2} = \frac{\sqrt{\pi}}{8} [3 + (6 + 2\beta^2)\beta^2] [\operatorname{erf}(a_{-}) - \operatorname{erf}(a_{+})]
\]

\[
- S_{b}(a_{+}) e^{-a_{+}^2} + S_{b}(a_{-}) e^{-a_{-}^2},
\]

with

\[
S_{b}(a) = \frac{1}{2} a^2 + \sqrt{2}b a^2 + \frac{a}{4 a(1 + 2\beta^2) + \frac{\beta}{\sqrt{2}}(2 + \beta^2)},
\]

\[
S_{2,1} = \frac{\sqrt{\pi}}{2} \operatorname{erf}(\sqrt{2}a) - \sqrt{2} a e^{-2a^2},
\]

\[
S_{2,2} = \frac{\sqrt{\pi}}{2} \{\operatorname{erf}(\sqrt{2}a) - \sqrt{2}\alpha e^{-2a^2}\}.
\]

The analytical expression of the kurtosis factor
for the non–truncated case is deduced readily from Eq. (15) by letting $\alpha$ to infinite and taking into account Eq. (13). It reads as

$$k = \frac{(1 + \exp(\beta^2/2))[(3 + \sqrt{3} + 6\beta^2 + \beta^3)\exp(\beta^2/2)]}{[1 + (1 + \beta^2\exp(\beta^2/2))]^2}.$$  

(19)

Note that, the kurtosis factor depends in this case only on the parameter $\beta$.

5. Kurtosis factor of super-Gaussian beams

The field of a super-Gaussian beam in the waist plane $z = 0$ is given by [3]

$$E_0(x, 0) = A \exp(-\alpha x^2/\omega_0^2),$$  

(20)

where $\alpha$ is the super-Gaussian power and $\omega_0$ is a scalar factor. A is the maximum value of the field which is taken in the following equal to unity. For $s=2$, the field in Eq. (20) reduces to a Gaussian one whereas the profile becomes flatter when the order is higher.

Substituting Eq. (20) into Eq. (2) with using the expanding form of the exponential function and with the help of Eq. (8.354) of Ref. [15], one obtains the following closed-form expression of the kurtosis factor

$$k = \frac{\gamma(5/s, (\sqrt{2}a)^s)\gamma(1/s, (\sqrt{2}a)^s)}{[\gamma(3/s, (\sqrt{2}a)^s)]^2}.$$  

(21)

For $s=2$, by using Eqs. (6.5.16) and (6.5.22) of Ref. [14], Eq. (21) reduces to that of Gaussian beam (Eq. (14)).

The corresponding expression for the non-truncated case is obtained by letting $\alpha \to \infty$. In this case, the gamma incomplete function tends to the gamma function and then the kurtosis factor is expressed as

$$k = \frac{\Gamma(5/s)\Gamma(1/s)}{[\Gamma(3/s)]^2}.$$  

(22)

6. Numerical results and discussion

6.1 HChG beams

To illustrate the analytical expression of the kurtosis factor of HChG beams, numerical calculations were performed by using Eq. (7) for truncated and non-truncated cases.

6.1.1 Truncated case

In Figs. 1a, b, c and d, we present the kurtosis factor versus $\alpha$ with $\beta$ being a parameter, for two values of $m$ ($m=0$ and 1). As can be seen, the factor $k$ increases with increasing $\alpha$ for the two values of the mode index, and then approaches the constant values which correspond to those for the non-truncated case. The values of $k$ for $\beta=0$ are 3 and 1.664 for $m=0$ and 1, respectively. For $\beta=2$, $k=1.867$ and 1.366 for $m=0$ and 1, respectively.

In the case of $m = 0$, the HChG beam reduces to the cosh-Gaussian one and further if $\beta = 0$, the constant value of $k$ is 3 which corresponds to the mesokurtic mode (i.e. the pure Gaussian beam). When $\beta$ (or/and $m$) increases, the constant values decrease, i.e. the mode becomes platikurtic (flatter than Gaussian beam).

Figs. 2a and b illustrate the variation of the factor $k$ versus $\beta$ for various values of $\alpha$ ($\alpha = 0.8, 1, 1.5$ and 25) and for $m = 0$ and 1. As can be expected, the factor $k$ decreases with increasing $\beta$ or $m$. We note that some cross points appear when small values of $\alpha$ are chosen. It means that for this case, two HChG modes with different values of $\alpha$ have the same factor. Some cross points were also observed in the curves of $M_2^s$ factor versus $\alpha$ [10].

6.1.2 Non-truncated case

Fig. 3 presents the kurtosis factor variation versus the index mode $m$ for $\beta = 0, 1, 2$ and 4. It turns out that for all values of $\beta$, the factor $k$ decreases with increasing $m$ and then approaches the asymptotic values. This result can be explained by the repartition of intensity distribution between the lobes of the HChG beam when $m$ increases; the intensity in the outer lobes is greater than the centered ones.

For $\beta = 0$, we find exactly the behavior of $k$ versus $m$ for the Hermite-Gaussian beams described by Saghafi et al [12].

In Fig. 4, we illustrate some plots of $k$ versus the parameter $\beta$. From this last figure, we can determine the kurtosis factor of some particular cases as Hermite-Gaussian ($\beta=0$) and cosh-Gaussian beams ($m=0$).

6.2 Super-Gaussian beams

To illustrate the analytical expression of the kurtosis factor, we give its variation versus $\alpha$ for different values of $s$ in Fig. 5. It is shown that $k$ increases with increasing $\alpha$ and reaches constant values which correspond to the non-truncated cases.

For $s=2$, the curve coincides with that in Fig.1(a). When $s$ increases highly, for example $s=20$, the super-Gaussian beam becomes flatter and so the beam is expected to be platikurtic.
Figure 1: Plot of kurtosis factor $k$ versus $\alpha$ for various values of $m$ and $\beta$.

Figure 2: Variation of $k$ versus $\beta$ for various values of $\alpha$: (a) $m=0$ and (b) $m=1$. 
7. Conclusion

In this study, the closed–form expressions of the kurtosis factor of truncated HChG and super-Gaussian beams are derived and some numerical examples are illustrated. In the limiting cases, our results reduce to those of Gaussian, cosh-Gaussian and Hermite-Gaussian beams. The results of the non-truncated case are obtained when the truncation parameter approaches to infinite.

References