Fraunhofer diffraction by conical tracks

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Abstract

Within the framework of the Fraunhofer diffraction theory, we investigate the intensity characteristics in the far-field region for a diffractive conical aperture irradiated by a plane wave. This calculation is applied to diffraction by conical tracks. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Among the solid-state nuclear track detectors, the CR-39 is often used in various domains such as dosimetry [1–5]. The last, and one of the most important, stage of this type of investigation involves the observation of the alpha particles tracks with different energies that appear in the CR-39. Many automatic image processes aiming to count tracks are performed. In some cases, the tracks have a conical form. On the other hand, Fraunhofer diffraction by a conical aperture is an effect of a great practical significance in optical instrumentation.

In this paper, we derive from the diffraction theory an expression for the intensity patterns diffracted by a conical track. In our calculation the surface involved in the Huygens formula is limited to that of the cone. This corresponds to a situation where the planar part of the film surface would be masked, i.e. by a metallization at a grazing angle. In the case where the whole surface is transparent, the present diffraction amplitude interferes with that due to the rest of the planer surface, i.e. (apart from a δ-function at zero angle) the diffraction amplitude of a circular diaphragm. In contrast to the former amplitude, the latter one is obviously independent of z. Therefore this interference should provide a sensitive measurement of z, which is directly related to the energy of the particle.

2. Theory

Let us envision a typical arrangement: plane waves impinge on a thick screen Σ of index n2 containing a conical aperture (see Fig. 1) and the subsequent far-field diffraction pattern spreads across along a large distance to an observation screen σ. As known,
the expression for the optical disturbance at $P(q, \phi)$, arising from an arbitrary aperture in the far-field, is [6]

$$E \propto \frac{e^{i(\omega t - kR')}}{R'} \int_{\text{Aperture}} \mathbf{e}^{ik(Y + Z)/R'} dS,$$

(1)

where $R'$ is the distance from the aperture to the point $P$, $q$ is the radial distance on the observation plane and $k = 2\pi/\lambda$ is the wave number of the light.

For a conical aperture, symmetry would suggest introducing spherical polar coordinates in both the plane of the aperture and the plane of observation, as shown in Fig. 1. Therefore, let

$$z = r \cos \phi, \quad y = r \sin \phi,$$

(2a)

$$Z = q \cos \phi, \quad Y = q \sin \phi,$$

(2b)

and the differential element of area is

$$dS = rd\phi dr.$$

(3)

Because of the complete axial symmetry, the solution must be independent of $\phi$. Substituting these last expression into Eq. (1), and using the Bessel function of the first kind of order zero, the optical disturbance becomes

$$E \approx -j \frac{k}{R'} e^{ik\Delta},$$

(4)

where

$$I' = \int_{0}^{R} J_{0} \left( kr \right) e^{i(kq - 1)\Delta/r} r dr \approx\frac{1}{w^{2}k^{2}} \int_{0}^{r'} J_{0}(t) e^{-pt} dt,$$

(5)

with

$$w = \frac{q}{R'},$$

(6)

$$z' = kRw,$$

(7)

and

$$p = -j \frac{n_{2} - 1}{w \tan \alpha}.$$  

(8)

To get Eq. (4), we have used the fact that the phase of the incident light through the track at the point of coordinates $(x, y)$ is given by

$$\psi(x, y) = kn_{1} [z - \Delta(x, y)] + kn_{2} \Delta(x, y),$$

(9)

where $n_{1}$ ($= 1$) and $n_{2}$ are the refractive index of air and the material, respectively. Thus, the expression of the complex amplitude of a conical aperture, according to the depth, is a product of a Bessel function by an exponential function, both being dependent of the variable $r$. For that, expanding $I_{0}(t)$ into its equivalent infinite series and integrating the $n$th term, we get [7]

$$I_{n} = \frac{1}{(n!)^{2}} \left( -\frac{1}{2} \right)^{n} J_{n},$$

(10)

with

$$J = \int_{0}^{r'} t^{2n+1} e^{-pt} dt = \int_{0}^{1} u^{2n+1} e^{-pu} du.$$

(11)

Integrating by parts, we eventually get

$$w^{2}k^{2}I = \frac{1}{p^{2}} \sum_{n=0}^{\infty} \frac{1}{(n!)^{2}} \left[ -\left( \frac{1}{2p} \right)^{2} \right]^{n} \Gamma(2n + 2)$$

$$- \frac{z'}{p} e^{-pz'} \sum_{n=0}^{\infty} \frac{1}{(n!)^{2}} \left[ -\left( \frac{z'}{2} \right)^{2} \right]^{n}$$

$$\times \left[ 1 + \frac{2n + 1}{pz'} + \frac{(2n + 1)2n}{pz'^{2}} + \ldots ight]$$

$$+ \frac{(2n + 1)!}{(pz')^{2n+1}},$$

(12)

where $\Gamma$ is the Euler function.
By using the identities [7]

$$\frac{\Gamma(2n+2)}{n!} = \frac{2}{\sqrt{\pi}} 2nT(n + \frac{3}{2}),$$

(13)

and [8]

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \Gamma(n + \alpha + 1) = \frac{\Gamma(1 + \alpha)}{(1 - z)^{\alpha + 1}},$$

(14)

one obtains

$$w^2 k^2 I = \frac{1}{p^2} \left[ 1 + \frac{1}{p^2} \right]^{3/2} - \frac{2}{\sqrt{\pi}} \frac{1}{p^2} e^{-p^2} \times \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{1}{p^2} \right)^n \Gamma \left( n + \frac{3}{2} \right) e_{2n+1} \left( pc \right),$$

(15)

with

$$e_{2n+1} \left( pc \right) = \sum_{i=0}^{2n+1} \frac{(pc)^i}{i!}.$$

(16)

Making use of Eq. (8) and the identity

$$\sum_{i=0}^{2n+1} \frac{(pc)^i}{i!} = \alpha' + j \beta',$$

(17)

where

$$\alpha' = \sum_{i=0}^{n} \frac{(-1)^i p^{2i}}{(2i)!},$$

(18)

and

$$\beta' = \sum_{i=0}^{n} \frac{(-1)^{i+1} p^{2i+1}}{(2i+1)!},$$

(19)

with

$$p' = (n_z - 1) z.$$

(20)

Eq. (4) becomes

$$E = j \frac{e^{iKd}}{\kappa \Lambda_0} \left( a - be^{i\theta'} (\alpha'' + j \beta'') \right),$$

(21)

where

$$a = \frac{(n_z - 1) tg^2 \alpha}{\left( (n_z - 1)^2 - w^2 tg^2 \alpha \right)^{3/2}},$$

(22)

$$b = \frac{2 tg^2 \alpha}{\sqrt{\pi} (n_z - 1)^2},$$

(23)

$$\alpha'' = \sum_{n=0}^{\infty} \left[ \frac{w^2 tg^2 \alpha}{(n_z - 1)^2} \right]^n \frac{\Gamma \left( n + \frac{3}{2} \right)}{n!} \alpha',$$

(24)

and

$$\beta'' = \sum_{n=0}^{\infty} \frac{w^2 tg^2 \alpha}{(n_z - 1)^2} \frac{\Gamma \left( n + \frac{3}{2} \right)}{n!} \beta'.$$

(25)

The irradiance at point P is \( \langle (ReE)^2 \rangle \) or 1/2 \( EE^* \), that is [6]

$$2 EE^* = a^2 + b^2 (\alpha''^2 + \beta''^2)$$

$$- 2 ab (\alpha'' \cos p' - \beta'' \sin p'),$$

(26)

This equation is valid in the case if \( w^2 tg^2 \alpha < (n_z - 1)^2 \). In the case, \( w^2 tg^2 \alpha > (n_z - 1)^2 \), Eq. (26) becomes

$$2 EE^* = d^2 + b^2 (\alpha''^2 + \beta''^2)$$

$$- 2 ab (\alpha'' \cos p' + \beta'' \sin p'),$$

(27)

where

$$d' = \frac{(n_z - 1) tg^2 \alpha}{\left[ w^2 tg^2 \alpha - (n_z - 1)^2 \right]^{3/2}}.$$

(28)

3. Numerical calculations

From Eq. (26) (where \( w^2 tg^2 \alpha < (n_z - 1)^2 \)), we can perform the numerical calculation of the intensity in various situations. Fig. 2 shows the relative intensity \( I/I_0 \) (where \( I_0 \) is the intensity diffracted at \( \theta = 0 \)) diffracted by a conical track, with different diameters, as a function of the observation angle \( \theta \) given by \( \sin \theta = q/R \). The parameters used in the calculation of the relative irradiance are: \( \lambda = 0.632 \mu m \), \( n_z = 1.493 \) and \( z = 20 \mu m \). These parameters are typical of the CR-39 irradiated film by alpha
Fig. 2. Relative intensity diffracted by a conical track of a depth $z = 20 \, \mu m$ for different values of the track diameters.

Because of the axial symmetry, the towering central maximum corresponds to a high-irradiance circular spot. This central disk is surrounded by a dark ring. From Fig. 2, we find that, in the range of our calculation, when the aperture diameter $D$ approaches $\lambda$, the central disk can be very large indeed and the conical aperture begins to resemble a point source. This figure shows also that as $D$ increases, the secondary maximum increases in relative value.

In Fig. 3 we illustrate the relative intensity patterns calculated for a conical track, as a function of $kR \sin \theta$, for two values of the depth: $z = 10 \, \mu m$ and $z = 30 \, \mu m$. Fig. 3 shows that the relative intensity of the secondary maximum decreases when $z$ increases.

Figs. 4 and 5 show how the depth and the radius of the conical track affects the intensity pattern at the observation angle $\theta = 10^\circ$. An irradiance oscillation appears as a function of $z$ or $R$. These oscillations become especially important at small $z$ or large $R$.

In summary, we have determined in this paper the diffraction amplitude and the related intensity in the
Fraunhofer far-field regime, due to a conical track irradiated by an optical plane wave. The results show a sensitivity to both the radius $R$ of aperture and to the depth $z$ of the cone. Several special cases are discussed and illustrated by plots in the important cases. The choice of these plots, and their scaling, was made on the basis of providing useful information to those people dealing with typical problems in image processing. More generally, this theory may have future applications in optics.

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**References**